So far: L : s.s. Lie algebra

- T toral subalgebra $\rightsquigarrow$ $$L = \underset{\substack{\| \\ C_L(T)}}{L_0} \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \qquad (*)$$

- If T is maximal toral (Cartan), then $L_0 = C_L(T) = T$. so $(*)$

  becomes the Cartan decomposition $$L = \underset{\substack{\| \\ T}}{H} \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$
  Φ "root system"

Goal: get more info on Φ, L_α $(\alpha \in \Phi)$, and the interactions between the wt spaces (L_α, $H = L_0$).

Today: Finding "$\mathfrak{sl}_2$-triples" in L.

sl_2 - triples (Humphreys §8.3. Orthogonality Properties).

Prop. (a) Φ span H^*.

(b) If $\alpha \in \Phi$, then $-\alpha \in \Phi$.

(c) Let $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$, then

$$[xy] = \kappa(x,y)\boxed{t_\alpha} \neq 0$$

def. on next page.

(d) If $\alpha \in \Phi$, then $[L_\alpha, L_{-\alpha}]$ is one-dimensional w/ basis t_α.

(e). $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$. $\forall \alpha \in \Phi$.

(f). If $\alpha \in \Phi$, x_α is any nonzero elt in L_α, then there is an elt $y_\alpha \in L_{-\alpha}$ and an elt $h_\alpha \in H$ s.t. $\{x_\alpha, y_\alpha, h_\alpha\}$ is a sl_2-trip.

Preparation for the proof.

(1) (*). Def of t_α.

important def.

Recall that K is nondegenerate on $C_L(H)$, and $C_L(H) = H$, so $K|_H$ is nondegenerate.

Thus. $K|_H$ gives us an identification $\varphi : H \to H^*$ where $\varphi(x)$ is the map

$$\varphi(x) : h \mapsto K(x, h)$$

(Recall that φ is inj, hence an iso, since $K|_H$ is nondegenerate.)

Point: $K(t_\alpha, h) = \alpha(h)\ \forall h \in H$.

So, for every $\alpha \in H^*$, $\exists!$ elt x, st $\alpha(h) = K(x, h)\ \forall h \in H$. We define this x to be t_α.

"10.05 notes"

(2). Recall $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ $\forall \alpha, \beta \in \Phi$. As a consequence, $K(L_\alpha, L_\beta) = 0$ whenever $\alpha + \beta \neq 0$.

$K|_{L_0}$ is nondegenerate. A similar argument shows that $[L_\alpha, L_{-\alpha}] \neq 0\ \forall \alpha \in \Phi$, if $-\alpha \in \Phi$.

Pf of Prop: (1a) Φ spans H^*.

Pf: Use "duality" (the map $\varphi : H \to H^*$, $x \mapsto \kappa(x, -)$):

If Φ fails to span H^*, then $\{ t_\alpha \mid \alpha \in \Phi \}$ fails to span H.

Let $H_\Phi = \operatorname{Span}\{ t_\alpha \mid \alpha \in \Phi \}$. Then $H_\Phi \subsetneq H$, so $\exists\, k \neq 0$ s.t.

$k \perp H_\Phi$ (take any k in $H_\Phi^\perp$). But then "$\overset{\alpha(h)}{\kappa(t_\alpha, h)} = 0$ for all $\alpha \in \operatorname{Span}\Phi$, so

$[h, L_\alpha] = \{ \alpha(h) x : x \in L_\alpha \} = 0 \ \forall \alpha \in \Phi$.

It follows that $[h, L] = 0$, so $Z(L) \neq 0$. which cannot happen.

(b). If $\alpha \in \Phi$, then $-\alpha \in \Phi$.

Pf: if $-\alpha \notin \Phi$, then $L_\alpha \perp L_\beta \ \forall \beta \in \Phi$ and $L_\alpha \perp L_0$,

so $L_\alpha \perp L$, contradicting the nondeg. of κ_L.

(c). Let $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$, then $[xy] = \kappa(x,y)\, t_\alpha$.

Pf: $\kappa([xy], h) = \kappa(x, [yh]) = \kappa(x, \alpha(h) y) = \kappa(x,y)\,\alpha(h) = \kappa(x,y)\,\kappa(t_\alpha, h)$

$\left([yh] = -[hy] = -(-\alpha)(h)\, y \right)$

$= \kappa(\kappa(x,y)\, t_\alpha, h) \ \forall h \in H$.

So since $\kappa|_H$ is nondegenerate, $[xy] = \kappa(x,y)\, t_\alpha$.

similar

(d) If $\alpha \in \Phi$, then $[L_\alpha, L_{-\alpha}]$ is one-dimensional, with basis t_α.

Pf: We know $t_\alpha \neq 0$ and $\langle t_\alpha \rangle \supseteq [L_\alpha, L_{-\alpha}]$ by (c). So it suffices to know $[L_\alpha, L_{-\alpha}] \neq 0$.

use nondeg of κ again.

(e). $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.

Pf: Otherwise $[t_\alpha x] = \alpha(t_\alpha)x = 0$ and $[t_\alpha, y] = -\alpha(t_\alpha)y = 0$

$\forall x \in L_\alpha, y \in L_{-\alpha}$. Pick $x \in L_\alpha$, $y \in L_{-\alpha}$ so that $\kappa(x, y) = 1$.

Then by (c) $[xy] = \kappa(x,y)t_\alpha = t_\alpha$. Take

$S = \langle x, y, t_\alpha \rangle$. Then by explicit multiplication table computation

$[\cdot,\cdot]$	x	y	t_α
x	0	t_α	0
y	$-t_\alpha$	0	0
t_α	0	0	0

it's easy to check that S is solvable.

Moreover, we have $t_\alpha \in [S, S]$.

$\left.\begin{array}{l} \mathrm{ad}_L t_\alpha \text{ is nilp} \\ \mathrm{ad}_L t_\alpha \text{ is s.s since } t_\alpha \in H \end{array}\right\} \mathrm{ad}_L t_\alpha = 0$, $t_\alpha \in Z(L)$.

$\rightarrow\leftarrow$.

(f). If $\alpha \in \Phi$ and $x_\alpha \in L_\alpha$, then $\exists\, y_\alpha \in L_{-\alpha}$, $h_\alpha \in H$ s.t.

$\{x_\alpha, y_\alpha, h_\alpha\}$ is an sl_2-triple in the sense that

① $[x_\alpha, y_\alpha] = h_\alpha$, ② $[h_\alpha, x_\alpha] = 2x_\alpha$, ③ $[h_\alpha, y_\alpha] = -2y_\alpha$.

✓ by def. ✓ by def of y_α. ?

Pf: To find y_α is a matter of normalization: for our desired $y_\alpha \in L_{-\alpha}$,

by (c) we have $\underbrace{[x_\alpha, y_\alpha]}_{\text{want this to be } h_\alpha} = K(x_\alpha, y_\alpha)\, t_\alpha$.

We hope $[K(x_\alpha, y_\alpha) t_\alpha, x_\alpha] = 2x_\alpha$ in light of ②.

But $[K(x_\alpha, y_\alpha) t_\alpha, x_\alpha] = \underline{K(x_\alpha, y_\alpha)\, \alpha(t_\alpha)}\, x_\alpha$, so we need

$K(x_\alpha, y_\alpha)\, \alpha(t_\alpha) = 2$. This can be acheived since x_α is fixed and $\alpha(t_\alpha) \neq 0$.

(pick any $y \in L_{-\alpha}$, then rescale y to get y_α)

Let y_α be the unique elt in $L_{-\alpha}$ s.t.

$$K(x_\alpha, y_\alpha)\,\alpha(t_\alpha) = 2.$$

"Any x_α leads to the same h_α!"

Define $[h_\alpha] = [x_\alpha, y_\alpha]$. Then ① and ② hold.

$\boxed{\text{Note: while } x_\alpha \in L_\alpha \text{ is arbitrary and } y_\alpha \text{ depends on } x_\alpha,\ h_\alpha \text{ is independent of the choice for } x_\alpha\text{!}}$

It remains to check ③:

(a) Note that $h_\alpha = [x_\alpha, y_\alpha] \overset{(c)}{=\!=} K(x_\alpha, y_\alpha)\, t_\alpha$

$$\boxed{h_\alpha = \frac{2}{\alpha(t_\alpha)}\, t_\alpha = \frac{2t_\alpha}{K(t_\alpha, t_\alpha)}}.$$

(b) It follows that

$$[h_\alpha, y_\alpha] = \left[\frac{2t_\alpha}{K(t_\alpha, t_\alpha)}, y_\alpha\right] = \frac{2}{\cancel{K(t_\alpha, t_\alpha)}} \cdot \left(-\cancel{\alpha(t_\alpha)}\right) y_\alpha = -2y_\alpha. \quad \square$$