Last time,

- Toral subalgebras are automatically abelian.
(every elt is s.s.)

- Given any toral subalgebra T. we get

$$(*). \quad L = \underset{C_L(T)}{\underbrace{L_0}} \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where $L_\alpha = \{ v \in L : x \cdot v = \alpha(x) \cdot v \ \forall x \in T \}$

- There's motivation for T to be larger → Def: CSA.

Def: A Cartan subalgebra is a maximal toral subalgebra.

Today.

Prove ①.

Fact: ① If T is a Cartan subalgebra H, then $H = C_L(H)$

② When $T = H$ is Cartan, $(*)$ is called the Cartan decomp and enjoys extra nice properties: e.g. $\dim L_\alpha = 1 \ \forall \alpha \in \Phi$.

Pf that $H = C_L(H)$ for a Cartan subalgebra H:

Recall: Since H is toral, we have

$$L = \overset{\overset{C_L(H)}{\parallel}}{L_0} \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

$$\overset{\cup\!\!\mid}{H}$$

(a) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$

(b) If $\alpha+\beta \neq 0$, then $\kappa(L_\alpha, L_\beta) = 0$.

(c) The restriction of κ to L_0 is non degenerate.

Pf of $H = C_L(H)$:

We'll follow Humphreys Prop 8.2 and prove the following claims. Let $C = C_L(H)$

① C contains the ss and nilp part of all of its elts, i.e. if $x = x_s + x_n$ for $x \in C$, then $x_s \in C$ and $x_n \in C$.

pf: $x \in C \Rightarrow \operatorname{ad} x(h) = 0$ $\forall h \in H$. But $\operatorname{ad} x_s$, $\operatorname{ad} x_n$ are poly in $\operatorname{ad} x$, so $\operatorname{ad} x_s(h) = \operatorname{ad} x_n(h) = 0$ $\forall h \in H$, so $x_s, x_n \in C$.

② All s.s. elts of C lie in H.

Pf: Say $x = x_s \in C$. then $[x, H] = 0$, so $\langle H, x \rangle$ is toral.

By maximality of H, $\langle H, x \rangle = H$. i.e., $x \in H$.

③ $K|_H$ is nondegenerate

Pf: Say $x \in H$ is an elt s.t. $K(x, H) = 0$. We'll show $K(x, \underset{L_0}{C}) = 0$

and then invoke the nondegeneracy of $K|_C$ to conclude $x = 0$.

Take $y \in C$. Then $y_n \in C$ by ①. We'll show $K(x, y_n) = 0$

$$K(x, y_n) = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y_n) = 0$$

$$\left.\begin{array}{l}[\operatorname{ad} x, \operatorname{ad} y_n] = \operatorname{ad}[x, y_n] = \operatorname{ad} 0 = 0 \\ \operatorname{ad} y_n \text{ is nilp}\end{array}\right\} \Rightarrow \operatorname{ad} x \operatorname{ad} y_n \text{ is nilp}$$

But then $\overset{②}{y_s \in H}$

$K(x, y) = K(x, y_s) + K(x, y_n)$

$= 0 + 0$.

④ C is nilp.

Pf: By Engel's Thm, it suffices to show that $\operatorname{ad} x : C \to C$ is nilp for all $x \in C$. This is clear if x is nilp or if $\underbrace{x \text{ is semisimple}}_{\substack{①② \\ x \in H,\ [H, C] = 0}}$. In general, $x = x_s + x_n$ where x_s, x_n commute, so x is nilp.

⑤ $H \cap [C, C] = 0$

Pf: $K(H, [C, C]) = K([H, C], C) = K(0, C) \subseteq \{0\}$.

Since $H, [C, C] \subseteq C$ and K is nondegenerate on H, so $H \cap [C, C] = 0$.

⑥ C is abelian.

$C \supseteq \underbrace{[C,C]}_{C'} \supseteq [C,C'] \supseteq \dots \supseteq 0$

Pf: If not, $[C,C] \neq 0$. $\underbrace{C \text{ is nilp}}$

$\Downarrow$ (1)

$\exists\, 0 \neq z \in Z(C) \cap [C,C]$.

By ⑤, $z \notin H$, so z is not s.s. by ②. Thus, $z_n \neq 0$, but z_n is a poly w/ no constant term in z, so $z_n \in Z(C)$ since $z \in Z(C)$.

This means $K_C(z_n, x) = 0$ since $\mathrm{ad}_C z_n = 0$. So $z_n = 0$ by nondeg of K_C. Contradiction.

⑦ $C = H$. Pf: If not, pick $x \in C \setminus H$ with $x = x_n \neq 0$. Then by ⑥, $K_C(x,y) = \mathrm{tr}(\underbrace{\overset{\text{nilp}}{\mathrm{ad}_C x}\ \mathrm{ad}_C y}_{\text{commute}}) = 0\ \forall y \in C$, so $x = 0$ since K_C is non-deg. $\times$ $\square$