Last time: "Application of A.J.D" — weight spaces:

Given any abelian toral subalgebra T of a s.s. Lie algebra L,
(every elt is s.s.)

any L-module V decomposes as

$$V = \bigoplus L_\alpha \qquad L_\alpha := \{ v \in V : x \cdot v = \alpha(x) v \ \forall x \in T \}$$

"wt space", α a weight if $L_\alpha \neq 0$.

Special case: ad rep $L \circlearrowright L$

$$\rightsquigarrow \quad L = \underbrace{L_0}_{\substack{= C_L(T) \\ = T}} \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

Φ = nonzero wts

$L_0 = \{ v \in L : x \cdot v = 0 \ \forall x \in T \}$
$= C_L(T)$

Nice properties that will generalize:

① $T = L_0 = C_L(T)$.

② $\dim L_\alpha = 1 \ \forall \alpha \in \Phi$.

For $L = \mathfrak{sl}_n$, $T =$ "the diagonals", $\Phi = \{ \varepsilon_i - \varepsilon_j \mid 1 \le i, j \le n, \ i \neq j \}$, $L_{\varepsilon_i - \varepsilon_j} = \langle e_{ij} \rangle$

Today. More on the decomposition

$$(*) \qquad L = \underset{\substack{\shortparallel \\ C_L(T)}}{L_0} \oplus \bigoplus_{\alpha\in\Phi} L_\alpha \qquad \text{for } T, L, \text{ad}.$$

(1). Toral subalgebras are automatically abelian (so assuming T is toral is enough)

Pf: Let T be toral. Take $x \in T$. Then $\text{ad}_T x$ is diagonalizable.

Now let $y \in T$ be a eigenvector of ad x. Then $[x, y] = \lambda y$.

It suffices to show that $\lambda = 0$.

Decompose x into $\text{ad}_T y$ – eigenbasis: $x = x_1 + \cdots + x_n$, say $[y, x_i] = \lambda_i x_i$ $\forall i$.

Then $-\lambda y = [y, x] = \sum \lambda_i x_i$, so $[y, [y,x]] = [y, -\lambda y] = 0 = \sum \lambda_i^2 x_i \Rightarrow \lambda_i = 0\ \forall i \Rightarrow [x, y] = 0$.

□

(2). (EW. Lemma 10.1) In the decomposition $(*)$, for all $\alpha, \beta \in H^*$,

$(H = T)$

(a) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$

(b) If $\alpha+\beta \neq 0$, then $K(L_\alpha, L_\beta) = 0$.

(c) The restriction of K to L_0 is non degenerate.

Pf: (a). Let $x \in L_\alpha$, $y \in L_\beta$. Then $[h, x] = \alpha(h)x$, $[h,y] = \beta(h)y$ $\forall h \in H$.

Thus, $[h, [x,y]] = [[h,x], y] + [x,[h,y]] = \alpha(h)[x,y] + \beta(h)[x,y]$

$= (\alpha+\beta)(h) \cdot [x,y]$

so $[x,y] \in L_{\alpha+\beta}$.

(b). Since $\alpha+\beta \neq 0$, $\exists h \in H$ s.t. $(\alpha+\beta)(h) \neq 0$. Let $x \in L_\alpha$, $y \in L_\beta$, then

$$\alpha(h) K(x,y) = K(\alpha(h)x, y) = K([h,x], y) = -K([x,h], y) \overset{\text{invariance}}{=\!=} -K(x,[h,y])$$

$$= -K(x, \beta(h)y)) = -\beta(h) K(x,y) \implies K(x,y) = 0.$$

(c). This is a corollary of (b). Take $z \in L_0$. If $z \in L_0^{\perp}$, then $K(z, x) = 0$ $\forall x \in L_0$. By (b), we also have $K(z, y) = 0$ $\forall y \in \alpha$ where α is a nonzero wt. But then $z \in L^{\perp}$. So $z = 0$ since K is nondegenerate on L. $\square$

(3) For the decomposition $L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ to be refined enough (it's coarse if T is too small: e.g. sl_n, n large, can always take $T = \langle e_{11} - e_{22} \rangle$, then we'd have $T \subsetneqq C_L(T) = L_0$; HW: EW. EX 10.2.)

We want T to be large, ideally w/ $T = C_L(T) = L_0$. This motivates:

Def. A <u>Cartan subalgebra</u> in L is a maximal toral subalgebra.

Prop: (next time). If H is a Cartan subalgebra of L, then $H = C_L(H)$.

More definitions Cartan Decomposition. When T is a Cartan subalgebra H, the decomposition (**) $L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ is called the Cartan decomposition of L with respect to H. (Fact: The decomp. (**) satisfies $L_0 = C_L(H) = H$ and $\dim L_\alpha = 1$ $\forall \alpha \in \Phi$.)

The set Φ is called the root system of L (w.r.t. H).

E.g. We worked out the matrix realization of the classical Lie algebra of type C_ℓ from a bilinear form w/ Gram matrix $G = \left(\begin{array}{c|c} 0 & I_\ell \\ \hline -I_\ell & 0 \end{array}\right)$ (Lecture 2).

$\downarrow$

$C_\ell = \left\{ \begin{bmatrix} m & n \\ p & q \end{bmatrix} : p = p^t, n = n^t, m^t = -q \right\}$. The diagonal matrices in C_ℓ turns out to form a Cartan subalgebra. Ex: Work out the Cartan decomp.