Last time: - Every $x \in L$, L s.s, has a unique A.J.D.

$$x = d + n \quad \text{s.t}$$

(1) $\operatorname{ad} d$ is diag (2) $\operatorname{ad} n$ is nilp (3) $[d, n] = 0$.

- If $L \subseteq gl(V)$ for some V, then

$$A.J.D(x) = C.J.D(x) \quad \forall x \in L.$$

Today. I. Final result in Ch 9. (EW).

Thm 1. L s.s. $\theta : L \to gl(V)$ rep of L. $x \in L$.

Suppose $x = d + n$ is the AJD of x.

Then the C.J.D of $\underbrace{\theta(x)}_{\subseteq \operatorname{Im}\theta} \in gl(V)$ is $\theta(x) = \boxed{\theta(d)} + \theta(n)$.

the action of d on V is always diag.

Pf. By restricting to the image of θ, we may assume θ is surj.

Now $\operatorname{Im}\theta$ $(= GL(V)) \cong L/\ker\theta$, so $\operatorname{Im}\theta$ is s.s. and

AJD $(\theta(x))$ makes sense. Moreover, $\mathrm{AJD}(\theta(x)) = \mathrm{cJD}(\theta(x))$.

(Ex). <u>Fact:</u> L_1, L_2 semisimple. $\theta: L_1 \to L_2$ surj.

$$\mathrm{AJD}(x) = d+n \Longrightarrow \mathrm{AJD}(\theta(x)) = \theta(d)+\theta(n)$$

The desired conclusion now follows.

II. Application of Thm 9.16.

eigen<u>functions</u> for "d".

s.s. Lie algebras via roots

reps of s.s. Lie algebras via wts

Henceforth in the course, we fix $k = \mathbb{C}$.

Recall from linear algebra: (EW. 16.3.2.)

Let V be f.d.v.s. A collection of diagonalizable maps in $\mathrm{End}(V)$ are <u>simultaneously diagonalizable</u>, i.e., $\exists$ a basis of V which is an eigenbasis for all the maps in the collection, if they pairwise commute.

To use Thm 1 and the above fact, consider

Def: A s.s. Lie algebra is <u>toral</u> if all elts of it are s.s.

<u>Note:</u> Every s.s. Lie algebra L has a toral subalgebra.

<u>Reason:</u> We can't have x is nilp $\forall x \in L$, otherwise L is nilp, hence solvable, by Engel's Thm. Take $x = d + n$, $n \neq 0$, then $d \in L$, so $T = \langle d \rangle$ is toral. □

Now let L be a s.s. Lie algebra and let H be a subalgebra of L that ① is toral and ② is abelian.

(later: ① $\Rightarrow$ ②, so we may just assume H is toral)

Then for any rep $L \to \mathfrak{gl}(V)$, we may consider the restriction

$$\begin{tikzcd} H \arrow[r] \arrow[dr] & \mathfrak{gl}(V) \\ & L \arrow[u] \end{tikzcd}$$

$H \to \mathfrak{gl}(V)$. Since H is toral, every elt $h \in H$ acts on V diagonalizably by Thm 1. Since H is abelian, V must have a basis β s.t. $\forall v \in \beta$,

$$h \cdot v \overset{= \alpha(h)\cdot v}{=} \alpha_h \cdot v \quad \forall h \in H.$$

Note that the dependence $h \mapsto \alpha_h$ is linear. (e.g. $(2h)\cdot v = \alpha_{2h} \cdot v$, $= 2(h\cdot v) = 2\alpha_h \cdot v$ $\}$ so $\alpha_{2h} = 2\alpha_h$ ......)

So we'll write $\alpha(h)$ for α_h and view α as linear functional in H^*.

Conversely, given any $\alpha \in H^*$, it makes sense to consider

$$L_\alpha := \{ x \in V : h \cdot x = \alpha(h) x \quad \forall h \in H \}.$$

We call L_α a weight space of V; it's often trivial. When $L_\alpha \neq 0$, we say α is a weight.

Note: Since V has a simult. eigenbasis β for the H-action, each $v \in \beta$ gives rise to a nonempty wt space, and we have

$$V = \bigoplus_{\alpha : L_\alpha \neq 0} L_\alpha$$

$$L_\alpha := \{ x \in V : h\cdot x = \alpha(h)\, x \quad \forall h \in H \}.$$

E.g. (sl_2) $\quad L = sl_2 = \langle e, f, h \rangle \qquad H = \langle h \rangle.$

$$V = L = \overset{L_0}{\underset{\langle h\rangle}{\underset{\shortparallel}{L_\alpha}}} \oplus \underset{\langle e\rangle}{\underset{\shortparallel}{L_{\alpha'}}} \oplus \underset{\langle f\rangle}{\underset{\shortparallel}{L_{\alpha''}}}$$

Consider $\quad ad : L \to gl(L).$ ✓

$h\cdot h = [h,h] = 0\cdot h, \quad h\cdot e = [h,e] = 2e, \quad h\cdot f = [h,f] = -2f$

$\beta = \{ e, f, h \}$ is a simult. e-basis for $H = \langle h \rangle$.

A functional $\alpha : H \to k$ can be identified with the scalar $\alpha(h)$.

Now for $\alpha = 0$, i.e., α s.t. $\alpha(h) = 0$. we have $h\cdot h = \alpha(h)\cdot h$

for $\alpha' = 2$, i.e., α' s.t. $\alpha'(h) = 2$, we have $h\cdot e = \alpha'(h)\cdot e$.

for $\alpha'' = -2$. - — - - — — . $h\cdot f = \alpha''(h)\cdot f$

E.g. (sl_n). $L = sl_n$ $\quad H = \langle \underbrace{e_{ii} - e_{i+1,i+1}}_{\text{"}h_i\text{"}} : 1 \le i \le n-1 \rangle$. — diag.

$$\mathrel{\|}$$

$$\underbrace{\langle e_{ii} - e_{i+1,i+1} \rangle \sqcup \langle e_{ij} : i \neq j,\ 1 \le i, j \le n \rangle}.$$

$L \circlearrowleft_{ad} L = V$

Claim: These elts form a simultaneous eigenbasis for H:

Take $h \in H$ (so h is diagonal) take $x_i = e_{ii} - e_{i+1,i+1}$ or $x_{ij} = e_{ij}$, $i \neq j$.

$$L = \underbrace{L_0}_{=H} \oplus \bigoplus_{i \neq j} L_{\varepsilon_i - \varepsilon_j}$$

$h \cdot x_i = [\underbrace{h}_{\text{diag.}}, x_i] = 0 \cdot x_i$ so $x_i \in L_0 = \{x \in V \mid h \cdot x = 0(h) \cdot x \ \forall h \in H\}$.

$h \cdot x_{ij} = [h, e_{ij}] = he_{ij} - e_{ij}h \overset{h = \mathrm{diag}(a_1, \dots, a_n)}{=\!=\!=} a_i e_{ij} - a_j e_{ij}$

$= (a_i - a_j) \cdot e_{ij} = (\varepsilon_i - \varepsilon_j)(h) \cdot e_{ij}$.

so $x_{ij} \in L_{\varepsilon_i - \varepsilon_j}$

where $\varepsilon_i \in H^*$ is the function s.t. $\varepsilon_i(\mathrm{diag}(a_1, \dots, a_n)) = a_i$.