Last time: (Always work over $k=\bar{k}$, char $k=0$ for today).

Results on s.s. Lie algebras

① $I \subseteq L$ ideal. L is s.s $\Longrightarrow$ I is s.s. $L = I \oplus I^{\perp}$.

② L is s.s $\Longleftrightarrow$ $\exists$ simple ideals $L_1, L_2, \ldots, L_r$ s.t

$$L = \underbrace{L_1 \oplus \cdots \oplus L_r}_{\text{every simple ideal of } L \text{ equals some } L_j.}$$

2.5 EW. Lemma 9.12

L s.s., $I \subseteq L$ ideal $\Rightarrow$ $L/I \cong I^{\perp}$, s.s.

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③ L. s.s $\Longrightarrow$ $\operatorname{ad} L = \operatorname{Der} L$ (to be finished)

Also today. Abstract Jordan Decomposition (EW) $\to$ A.J.D.

By the result ③, we are done w/ the fundamental concepts of Lie algebra. Roughly, since Rad L is solvable and L/Rad L is s.s. for any Lie algebra L, we tried to get structural results on solvable (& nilp) and semisimple algebras

Engel's & Lie's Thms

Cartan's Criteria

Next: {
I. finer structure of s.s. algebras, via "roots" and the adj.

roots = wts of ad.

II. structure of reps of s.s. algebras, via "weights"

need A.J.D. and Weyl's Complete Reducibility Thm.

Pf of ③ $\quad$ (L s.s $\Rightarrow$ $\mathrm{Der}\,L = \mathrm{ad}\,L$)

(Recall: L s.s $\Rightarrow$ $\ker \mathrm{ad} = Z(L) = 0$ $\Rightarrow$ ad is inj. $\underset{\text{s.s.}}{\underset{\downarrow}{\mathrm{ad}\,L}} \cong L$.

$\mathrm{ad}\,L$ is an ideal in $\mathrm{Der}\,L$)

Let $M = \mathrm{ad}\,L \subseteq \mathrm{Der}\,L$. Then since $\mathrm{ad}\,L$ is an ideal of $\mathrm{Der}\,L$,

$\mathrm{Der}\,L = M \oplus M^\perp$. We'll show that $M^\perp = 0$.

Since M is s.s. K_M is nondegenerate so $M \cap M^\perp = 0$, hence

$[M, M^\perp] = 0$. Now if δ is an elt in $M^\perp$, then $[\underset{M^\perp}{\underset{\Uparrow}{\underline{\delta}}}, \underset{M}{\underset{\Uparrow}{\underline{\mathrm{ad}\,x}}}] = 0 \ \forall x \in L$.

But $[\delta, \mathrm{ad}\,x] = \mathrm{ad}\,\delta(x)$, so $\mathrm{ad}\,\delta(x) = 0 \ \forall x \in L$, so $\delta(x) = 0 \ \forall x \in L$.

so $\delta = 0$. $\square$

Abstract Jordan Decomposition.

Prop 1. L s.s. $\delta \in \mathrm{Der} L \subseteq \mathfrak{gl}(L)$.

Suppose $\delta = \underbrace{\sigma}_{\text{diag.}} + \underbrace{\nu}_{\text{nilp}}$ is the (concrete) Jordan decomp of δ.

then σ, ν are both derivations of L as well.

Helpful Lemma. $\delta \in \mathrm{Der} L$, $\lambda, \mu \in k$, $x, y \in L$, then

"Leibniz" HW1. Ex. $$(\delta - (\lambda+\mu) 1_L)^n [x, y] = \sum_{k=0}^{n} \binom{n}{k} [(\delta - \lambda 1_L)^k x, (\delta - \mu 1_L)^{n-k} y].$$

Pf of Prop 1. For $\lambda \in k$, let $L_\lambda = \{ x \in L : (\delta - \lambda 1_L)^m x = 0 \quad \forall m \gg 0 \}$.

generalized eigenspace of δ w/ e-value λ.

(e.g. $\quad JNF(\delta) = \left[\begin{array}{cc|ccc} 2 & 1 & & & \\ & 2 & & & \\ \hline & & \pi & 1 & \\ & & & \pi & 1 \\ & & & & \pi \end{array}\right]$ $\quad L_2$, L_π. $\quad L_\lambda = 0$ if λ is not an e-value for δ.)

By linear algebra, $\quad \underset{\delta \circlearrowleft}{L} = \bigoplus_{\lambda} L_\lambda$.

By the lemma $\quad (*)\quad [L_\lambda, L_\mu] = L_{\lambda+\mu} \quad \forall \lambda, \mu \in k$.

Note that since $\delta = \underset{\text{diag}}{\bar{\sigma}} + \nu$ is the c.J.D. of δ, the eigenspace of σ for λ is exactly L_λ. $\forall \lambda \in k$. By $(*)$, it follows that

$$\sigma\Big(\underbrace{[x,y]}_{\in L_{\lambda+\mu}}\Big) = (\lambda+\mu)\,[x,y]. \qquad \forall x \in L_\lambda,\ y \in L_\mu$$

On the other hand $[\sigma(x), y] + [x, \sigma(y)]$

$$= [\lambda x, y] + [x, \mu y]$$

$$= (\lambda + \mu)[x, y]. \qquad \forall x \in L_\lambda, y \in L_\mu.$$

Therefore σ is a derivation. Thus, $\gamma = \delta - \sigma$ is a derivation.

$\square$.

Thm. L s.s. Then each $x \in L$ can be written uniquely as a sum

Abstract Jordan Decomp.

$x = d + n$ where (1) d is ad-diagonalizable, (2) n is ad-nilp,

and (3) $[d, n] = 0$. Furthermore, if $[x, y] = 0$, then $[d, y] = [n, y] = 0$.

Def: The decomposition $x = d + n$ from the Prop is called the A.J.D. of x.

Pf: Need decomp $x = d + n$ s.t. (1) ✓ ad d is diag ✓ (2) ad n is nilp.

(3) $[d, n] = 0$. Now consider $\mathrm{ad}\, x \in \mathrm{Der}\, L \subseteq \mathfrak{gl}(L)$. It makes sense

to talk about the (concrete) J.D of $\mathrm{ad}\, x = \underset{\text{s.s.}}{\sigma} + \underset{\text{nilp.}}{\nu}$.

By Prop 1, $\sigma, \nu \in \mathrm{Der}\, L$. By ③, $\sigma = \mathrm{ad}\, d$ and $\nu = \mathrm{ad}\, n$

for some $\underline{d, n \in L}$ unique (ad is inj). But then we have (1) and (2). To see (3),

note that in the C.J.D. $0 = [\sigma, \nu] = [\mathrm{ad}\, d, \mathrm{ad}\, n] = \mathrm{ad}\, [d, n]$.

This implies $[d, n] = 0$ since ad is inj.

Furthermore, in the C.J.D., recall that ν is a poly of $\mathrm{ad}\, x$.

Say $\nu = p(\mathrm{ad}\, x)$ where $p = c_0 + c_1 X + \cdots + c_n X^n$. Now if $[x, y] = 0$, then $v(y) = [p(\mathrm{ad}\, x)](y) = c_0 y$. Since v is nilp. $c_0 = 0$.

So $v(y) = 0$, i.e., $\mathrm{ad}\, n(y) = [n, y] = 0$.

It follows that $[d, y] = [x, y] - [n, y] = 0 - 0 = 0$ □.

Note: For any Lie algebra L, we now have A.J.D.

$$x = d + n \quad \forall x \in L. \quad ①$$

If L is linear, i.e., $L \subseteq \mathfrak{gl}(V)$ for some V.

then x has C.J.D $\quad x = d + n \quad \forall x \in L$. ②

Q: Do they necessarily agree? A: Yes. since ② $\xRightarrow[\text{HW}]{\text{earlier}}$ CJD$(\mathrm{ad}\, x) = \mathrm{ad}\, d + \mathrm{ad}\, n$. □