Last time: Proved Cartan's Criteria

Throughout today's lecture, L is a Lie algebra / $k = \bar{k}$, char $k = 0$.

Cartan's Criteria: L is solvable $\iff$ $K(x, y) = 0$ $\forall x \in L,\ y \in L'$.

L is s.s $\iff$ K is non-degenerate.

Today. Applications of Cartan's Criteria: 3 results on s.s. Lie algebras

Def: Let $I_1, \dots, I_r$ be ideals in L. We say L is a direct sum of $I_1, \dots, I_r$ if $L = I_1 \oplus \cdots \oplus I_r$ as vector spaces.

Note: $i \neq j \Rightarrow [I_i, I_j] \subseteq I_i \cap I_j = 0$, so $L = I_1 \oplus \cdots \oplus I_r$ as Lie algebras.

"external direct sum"

Result 1. Prop. If L is s.s and I is a nontrivial proper ideal in L, then $L = I \oplus I^{\perp}$ and I is s.s. itself.

Pf: Let $\kappa = \kappa_L$ be the Killing form on L. Let $J = I \cap I^{\perp}$, then $\kappa_J = \kappa|_J$, and $\kappa_J(x,y) = \kappa(x,y) = 0 \quad \forall x, y \in J$.

So J is solvable by Cartan's first criterion. Therefore $\underline{J = I \cap I^{\perp} = 0}$. ①

On the other hand, $\underline{\dim I + \dim I^{\perp} = \dim L}$ ② (by bilinear algebra)

By ①, ②, we must have $L = I \oplus I^{\perp}$

We now show I is s.s. If not, then K_I must be degen. by Cartan's 2nd Criterion, i.e., $\exists\, a \in I$ s.t. $K_I(a,x) = 0 \ \forall x \in I$.

But $K_I = K|_I$, so $K(a,x) = 0 \ \forall x \in I$. Since $L = I \oplus I^{\perp}$,

$$\underset{z}{L} = \underset{x}{I} \oplus \underset{y}{I^{\perp}}, \quad z = x + y$$

It follows that $K(a,z) = 0 \ \forall z \in L$, so K is degenerate.

This contradicts the s.s. of L. $\square$

"s.s = direct sum of simples"

Result 2. Thm. L is s.s $\iff$ $\exists$ simple ideals $I_1, \dots, I_r$ s.t. $L = \bigoplus_{j=1}^{r} I_j$.

Moreover, if L is s.s. then every simple ideal I of L is equal to I_j for some j.

Pf: First assume L is s.s.

($\Rightarrow$) To find the ideals $I_1, \dots, I_r \subseteq L$ s.t. $L = \bigoplus I_j$.

Use induction on $\dim L$. Let I be an ideal of L w/ minimal possible dimension. Then I is simple by minimality.

If $I = L$, we are done. If $I \neq L$, then $L = I \oplus I^{\perp}$ by Result 1 where $\dim I^{\perp} < \dim L$. By induction, $\exists$ ideals $I_2, \dots, I_r$ of $I^{\perp}$ s.t. $I^{\perp} = I_2 \oplus \dots \oplus I_r$. But then $[L, I_j] \subseteq [I, I_j] \oplus [I^{\perp}, I_j] \subseteq I_j$. So I_j is an ideal of L $\forall\, 2 \leq j \leq r$. Setting $I_1 = I$, we get the desired $I_1, \dots, I_r$.

Let $I \subseteq L$ be a simple ideal. We now prove that $I = I_j$ for some $1 \leq j \leq r$. Note that $[I, L]$ is an ideal of I, so $[I, L] = I$ or $[I, L] = 0$. The latter is impossible since it implies $0 \neq I \subseteq Z(L) = 0$. So $[I, L] = I$. But then

$$I = [I, L] = \bigoplus_{j=1}^{r} [I, I_j].$$

Since I is simple, we must have $[I, I_j] = I$ for some j. This implies $I \subseteq I_j$, so $I = I_j$ since I_j is simple.

Finally we prove L is s.s if $L = I_1 \oplus \cdots \oplus I_r$, I_j simple $\forall j$.

Let $I = \operatorname{Rad} L$. We need $I = 0$. Note that $[I, I_j] \subseteq I \cap I_j$ is a solvable ideal of I_i, so $[I, I_j] = 0$, $\forall j$.

since I is solvable

Thus, $[I, L] \subseteq \oplus [I, I_j]$ is zero, i.e., $I \subseteq Z(L)$.

But $Z(L) = Z(L_1) \oplus \cdots \oplus Z(L_r) = 0$, so $I = 0$. $\square$

Result 3. Prop. If L is s.s, then $\underline{\mathrm{ad}\, L} = \mathrm{Der} L$.

inner

(All derivations of a s.s. Lie alg are inner.)

Recall: – $\mathrm{Der} L$ is a Lie algebra under $[\ ] =$ "commutator". (simply because L is a "k-algebra")

– $\mathrm{ad}\, x$ is a derivation $\forall x \in L$, so $\mathrm{ad} : L \to \mathfrak{gl}(V)$

has image $\mathrm{ad}\, L \subseteq \mathrm{Der} L \subseteq \mathfrak{gl}(V)$, and the prop just

says $\mathrm{ad} : L \to \mathrm{Der}\, L$ is surj.

Note: – $\mathrm{ad}\, L \subseteq \mathrm{Der} L$ is an ideal in $\mathrm{Der} L$: $[\delta, \mathrm{ad}\, x] = \mathrm{ad}(\delta x)\ \forall \delta \in \mathrm{Der} L$.

– Since L is s.s., $\ker \mathrm{ad} = Z(L) = 0$, so $\mathrm{ad}\, L \cong L$.

Pf: next time.