Last time: 1. Killing form on L $\qquad K(x,y) = \mathrm{tr}(\mathrm{ad}\,x\ \mathrm{ad}\,y)$

symm. bilinear, invariant ($K([xy],z) = K(x,[yz])$).

$I^\perp$ is an ideal if I is an ideal.

2. $x \in \underset{\in L \subseteq}{gl(V)}$ $\longrightarrow$ $\exists!$ decomp $x = d + n$ (J.D.) s.t.

(1) d is diagonalizable (aka. semisimple)

(2) n is nilp

(3) $[d,n] = 0$, i.e., d, n commute.

Ex: If $x \in L \subseteq gl(V)$ has J.D. $x = d+n$,

then $\mathrm{ad}\,x \in gl(L)$ has J.D. $\mathrm{ad}\,x = \underline{\mathrm{ad}\,d} + \underline{\mathrm{ad}\,n}$.

i.e., (1) $\mathrm{ad}\,d$ is diagonalizable, (2). $\mathrm{ad}\,n$ is nilp.

(3) $[\mathrm{ad}\,d, \mathrm{ad}\,n] = 0$.

Later: Thm ① For every Lie algebra L / $k=\bar{k}$ and $\forall x \in L$,

(abstract J.D.) $\exists$ a unique decomp. $x = d + n$ where

(1) d is s.s. (2) n is nilp. (3) $[d, n] = 0$.

②. (Abstract J.D.'s control J.D. in any L-rep)

Say x has abstract J.D. $x = d+n$ in L.

Then for all rep $\varphi : L \to \mathfrak{gl}(V)$ of L,

the J.D. of $\varphi(x)$ in $\mathfrak{gl}(V)$ must be

$$\varphi(x) = \varphi(d) + \varphi(n).$$

Important for rep. theory of L.

Today. Cartan's criteria for $\underbrace{\text{solvability}}_{\text{Rad}\, L = L}$ and $\underbrace{\text{semisimplicity}}_{\text{Rad}\, L = 0}$.

Thm. Let L be a Lie algebra / $\mathbb{C}$. Let K be the Killing form on L. Then

(1) L is solvable $\iff$ $K(x,y) = 0$ $\forall x \in L, y \in L' := [L,L]$.

(2) L is semisimple $\iff$ K is nondegenerate.

Pf: (1). By the notes of 9.18, it suffices to show the following:

Prop. If $L \subseteq \mathfrak{gl}(V)$, V f.d., and $\operatorname{tr}(xy) = 0$ $\forall x, y = 0$, then L is solvable.

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Pf: We'll prove that every elt $x \in L'$ is nilp. This implies that L' is nilp by Engel's Thm, so L' is solvable and L is solvable.

Let $x \in L'$ and let $x = d + n$ be its J.D. It suffices to prove that $d = 0$. Let β be the basis of V s.t.

$$[x]_\beta = JNF(x) = \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

n

d

Say $[d]_\beta = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

We'll show $d = 0$ by showing that $\sum_{i=1}^{n} \lambda_i \bar{\lambda}_i = 0$.

Let $\bar{d}$ be the linear map in $\mathfrak{gl}(V)$ s.t.

$$[\bar{d}]_\beta = \operatorname{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n).$$

Then $\sum \lambda_i \bar{\lambda}_i = \operatorname{tr} \bar{d} x$, so to show $\sum \lambda_i \bar{\lambda}_i = 0$

it further suffices to show that $\operatorname{tr} \bar{d} x = 0$. Since $x \in L'$,

it's enough to show that $\operatorname{tr} \bar{d}([yz]) = 0 \quad \forall y, z \in L$.

By the invariance property of tr, we have

$$\operatorname{tr}(\bar{d}[yz]) = \operatorname{tr}([\bar{d}y]z),$$

so we just need to show that $\operatorname{tr}(\underbrace{[\bar{d}y]}_{\in \mathfrak{gl}(V)}z) = 0 \quad \forall y, z \in L$.

(Recall that $\operatorname{tr}(y'z) = 0 \quad \forall y', z \in L$ by assumption)

To finish the proof, it now suffices to show that

$$[\bar{d}\, y] \in L \quad \forall y \in L.$$

To see that, recall from Ex. that $\operatorname{ad} x = \operatorname{ad} d + \operatorname{ad} n$ must be the J.D. of $\operatorname{ad} x \in \mathfrak{gl}(L)$, therefore $\exists\, q(x) \in \mathbb{C}[x]$ s.t.

$$\overline{\operatorname{ad} d} = q(\operatorname{ad} x)$$

But then $\operatorname{ad} \bar{d} = \overline{\operatorname{ad} d} = q(\operatorname{ad} x)$. Since $\operatorname{ad} x(y) \in L\ \forall y \in L$, $q(\operatorname{ad} x)(y) \in L\ \forall y \in L$. $\square$

(2). "L is semisimple $\Longleftrightarrow$ K is nondegenerate".

We'll prove " L has a nontrivial solvable ideal $\Longleftrightarrow$ $L^\perp \neq 0$".

($\Leftarrow$) Say $L^\perp \neq 0$. Then $L^\perp$ is nontrivial ideal of L.

For any $x \in L^\perp$, $y \in (L^\perp)' \subseteq L$, we have

$$K(x, y) = 0 ,$$

therefore by ①, $L^\perp$ is solvable. so L has $L^\perp$ as a nontrivial solvable ideal.

($\Rightarrow$). Say L has a nontrivial solvable ideal. Pick a minimal such ideal A. Then $[A, A] = 0$. i.e., A is abelian, by minimality, We claim that $A \subseteq L^{\perp}$.

To prove the claim, take $a \in A$, $x \in L$. Then

$$K(a,x) = \mathrm{tr}(\mathrm{ad}\, a \,\mathrm{ad}\, x).$$

But $(\mathrm{ad}\, a \,\mathrm{ad}\, x)^2(y) = \mathrm{ad}\, a \,\underbrace{\mathrm{ad}\, x \,\underbrace{\mathrm{ad}\, a \,\underbrace{\mathrm{ad}\, x \; y}_{L}}_{A}}_{A} = 0 \quad \forall y \in L$

So $\mathrm{ad}\, a \,\mathrm{ad}\, x$ is nilp. Nilp maps have trace zero, so $K(a,x) = 0$ $\forall a \in A$, $x \in L$. So $A \subseteq L^{\perp}$. $\square$