Last time: – Consequences of Engel's and Lie's Thms

– Stated Cartan's Solvability Criterion.

$L/k=\bar{k}$, char $k=0$ $\Rightarrow$ L is solvable iff

$\mathrm{tr}(\mathrm{ad}x\,\mathrm{ad}y)=0 \quad \forall x\in L, y\in L'$.

$L' = [L,L]$

– Reduced C.S.C. to

$\underline{L\subseteq \mathfrak{gl}(V)}/k=\bar{k}$, char $k=0$ $\Rightarrow$ L is solvable if

$\mathrm{tr}(xy)=0 \quad \forall x,y\in L$.

Today. Review of some linear algebra.

1. The Killing Form

Def: The Killing form on a Lie algebra is the symmetric bilinear form κ s.t.

$$\kappa(x,y)=\mathrm{tr}(\mathrm{ad}x\,\mathrm{ad}y) \quad \forall x,y\in L$$

Note: (1) Now C.S.C. can be stated as

- (under the same assumptions) L is solvable iff $K(x,y)=0$ $\forall x\in L, y\in L'$.

$\mathrm{Rad}\,L = L$

- Later: (Cartan's Semisimplicity Criterion) L is semisimple iff K is nondegenerate.

$\mathrm{Rad}\,L = 0$

(2) Note that K is indeed bilinear and symmetric. $K(x,y) = tr(\mathrm{ad}x\,\mathrm{ad}y)$.

bilinearity: follows from linearity of ad, composition, and tr.

Symmetric: follows from symmetry of trace $tr(ab) = tr(ba)$

(3). K is invariant in the sense that

(Similarly, if $L \in gl(V)$, then $tr([xy]z) = tr(x[yz])$)

$$K([xy], z) = K(x, [yz]) \overset{\text{similarly}}{=\!=} tr(\mathrm{ad}x\,\mathrm{ad}y\,\mathrm{ad}z - [\mathrm{ad}x\,\mathrm{ad}z]\mathrm{ad}y)$$

$$\| $$

$$tr(\mathrm{ad}[xy]\mathrm{ad}z) = tr([\mathrm{ad}x, \mathrm{ad}y]\mathrm{ad}z) = tr(\mathrm{ad}x\,\mathrm{ad}y\,\mathrm{ad}z - \mathrm{ad}y[\mathrm{ad}x\,\mathrm{ad}z])$$

Killing form and ideals

(1). Compatibility w/ ideals:

<u>Prop.</u> Let I be an ideal of a Lie algebra L. Let $K = K_L$ be the Killing form on L and let $K_I : I \times I \to k$ be the Killing form on I.

Then
$$\underset{\operatorname{tr}(\operatorname{ad}_I x \circ \operatorname{ad}_I y)}{K_I(x,y)} = \underset{\operatorname{tr}(\operatorname{ad}_L x \circ \operatorname{ad}_L y)}{K_L(x,y)}$$

<u>Pf:</u> Extend a basis γ of I to a basis $\beta = \gamma \sqcup \gamma'$ of L. Then $\forall x \in I$, since $[x, L] \subseteq I$, we have

$$[\operatorname{ad}_L x]_\beta = \begin{array}{c} \left[\begin{array}{c|c} A_x & B_x \\ \hline 0 & 0 \end{array}\right] \end{array}\begin{array}{c} \gamma \\ \gamma' \end{array}$$ for some A_x, B_x.

Consequently,

$$[\,\mathrm{ad}_L x \ \ \mathrm{ad}_L y\,]_\beta = \left[\begin{array}{c|c} A_x & B_x \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} A_y & B_y \\ \hline 0 & 0 \end{array}\right]$$

$$= \begin{array}{cc} \left[\begin{array}{c|c} A_xA_y & A_xB_y \\ \hline 0 & 0 \end{array}\right] & \begin{array}{c} \gamma \\ \gamma' \end{array} \end{array}$$

So $K_L(x,y) = \mathrm{tr}(A_xA_y) + 0 = \mathrm{tr}(\mathrm{ad}_I x \ \mathrm{ad}_I y) = K_I(x,y)$. $\square$

(2) Perpendicular space:

<u>Def.</u> For any subset $S \subseteq V$, V v.s. w/ b.l form β, the set

$$S^\perp = \{ x \in V : \beta(x,s) = 0 \quad \forall s \in S \}$$

is called the <u>perpendicular space</u> of S. We say β is <u>non-degenerate</u> if $V^\perp = \{0\}$ and <u>degenerate</u> otherwise.

Prop: Let I be an ideal in a Lie algebra L.

Then $I^\perp$ is also an ideal in L (w.r.t. κ).

Pf: Take $x \in L$, $y \in I^\perp$

$$\kappa([x,y], z) = -\kappa([y,x], z) = -\kappa(\underbrace{y}_{I^\perp}, \underbrace{[x,z]}_{I}) = 0 \quad \forall z \in I. \qquad \square$$

2. Jordan decomposition

Work over $k = \bar{k}$. (no assump. on char k needed).

<u>Linear Lie algebras.</u> Let $L \subseteq \mathfrak{gl}_k(V)$ be a Lie algebra.

<u>Recall:</u> (1) Every $x \in L \subseteq \mathfrak{gl}(V)$ has a Jordan normal form

$$JNF(x) = \begin{bmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \cdots & \\ & & & J_{\lambda_m} \end{bmatrix}$$

where each Jordan block is of the form

$$J_{\lambda_i} = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

The JNF is unique up to reordering of the blocks

The minimal polynomial of x is $\prod (x-\lambda)^{a_\lambda}$

over the λ_i appearing in $JNF(x)$, a_λ = max height of a λ-block.

eg. $JNF(x) = \left[\begin{array}{cc|cc|c} 2 & 1 & & & \\ & 2 & & & \\ \hline & & 2 & 1 & \\ & & & 2 & \\ \hline & & & & 3 \end{array}\right]$ $= V_2$ $\quad x-2I$

$V_3 \quad x-3I$

$\longrightarrow m(x) = (x-2)^3 (x-3)$

- It follows that x can be written as $x = d + n$ where d [(1)] is diagonalizable, [(2)] n is nilp., and [(3)] d and n commute (by the way block matrices multiply). Such a decomp. turns out to be unique: every $x \in L$ can be written uniquely as $x = d + n$ where (1), (2), (3) hold. It's called the Jordan decomposition of x.

- For the uniqueness proof, we need: (Lemma 16.6 [EW]).

CRT. Say the min poly of x is $\prod (x - \lambda)^{a_\lambda}$. Let $V = \bigoplus V_\lambda$ be the decomp of V onto the generalized eigenspaces of x.

Then if we choose $\mu_\lambda \in k$ for each λ, $\exists$ a poly $p(x)$ s.t. $p(x) = \bigoplus \mu_\lambda V_\lambda$. $\square$

Application: $\exists\ p(x) \in k[x]$ s.t. $p(x) = \bigoplus \lambda V_\lambda = d$.

i.e., d is a poly in x.

Consequently, $n = x - d$ is a polynomial in d.

Consequently, $[d, n] = 0$

Cor. The Jordan Decomp is unique.

Pf: Say $x = d' + n'$ is another decomp where d' is diag. n' is nilp. and $[d', n'] = 0$. Then $d + n = d' + n'$, so $d - d' = n - n'$. where d, d', n, n' pairwise commute. Thus, $d - d'$ is diagonalizable (since d, d' commute) while $n - n'$ is nilp. So $d - d' = n - n' = 0$. $\square$