Last time. We proved the two key Lemma's for Engel's and Lie's Thm.

Today.

I. Some corollaries of Engel's and Lie's Thm.

II. Cartan's Solvability Criterion.

I. Coro. of Engel's Thm:

Cor1 L nilp, $0 \neq K \subseteq L$ ideal $\Longrightarrow K \cap Z(L) \neq 0$

in particular, $Z(L) \neq 0$

Pf: Since K is an ideal, L acts on K by the adjoint action.

So we have a map $ad_K : L \longrightarrow gl(K)$.

"V"

$ad_K L$ "L"

Note that adx is nilp $\forall x \in L$ since x is nilp. So by Engel's Thm,

$0 \neq v \in K$ s.t. $ad_K(L) . v = 0 \Longrightarrow v \in K \cap Z(L)$. $\square$

Lie: Assume $k = \bar{k}$, $\operatorname{char} k = 0$.

Cor 2. Let L be solvable $/k$. Then $\exists$ a chain of ideals of L

$$0 = L_0 \subset L_1 \subset L_2 \cdots \subset L_n = L$$

s.t. $\dim L_i = i \quad \forall\, 0 \le i \le n$.

Pf: Consider the adjoint rep $\quad ad : L \to \underbrace{ad\, L}_{\text{"}L\text{"}} \subseteq gl(\underbrace{L}_{\text{"}V\text{"}})$

By Lie's Thm, $\exists$ a flag $\quad 0 = V_0 \subset V_1 \subset \cdots \subset V_n = V = L$

s.t. $\dim V_i = i \quad \forall\, 0 \le i \le n$. Take $L_i = V_i$. (Since

$ad\, L \cdot V_i \subseteq V_i \iff V_i$ is an ideal in L) $\quad \square$

Cor 3. L solvable $\Rightarrow [L, L]$ is nilp.

Pf: Consider the ideal chain $0 = L_0 \subset \overset{\circlearrowleft x}{L_1} \subset \cdots \subset L_n = L$ from Cor 2

Pick a basis for L, $\beta = \{v_0, v_1, \dots, v_n\}$ s.t.

$$L_i = \langle v_0, \dots, v_i \rangle \quad \forall\, 0 \le i \le n.$$

Then $\forall x \in L$, the matrix of $\operatorname{ad} x \in t(n, k)$ w.r.t. β.

So $\operatorname{ad} L \subseteq t(n, k)$, so $\operatorname{ad}[L, L] = [\operatorname{ad} L, \operatorname{ad} L] \subseteq n(n, k)$

$(\operatorname{ad}[x, y] = [\operatorname{ad} x, \operatorname{ad} y] \ \forall x, y \in L)$

So all elts of $[L, L]$ are ad-nilpotent. By Lie's Thm, it follows that $[L, L]$ is nilp. $\square$

Rmk: Note that we've often used the adjoint action

$$\mathrm{ad} : L \to \underset{\text{"}L\text{"}}{\mathrm{ad}\,L} \subseteq \mathfrak{gl}(\underset{\text{"}V\text{"}}{L})$$

to deduce information about general Lie algebras L from results on linear Lie algebra.

(One more instance:) Cartan's Solvability Criterion.

Prop 1. Let $L \subseteq \mathfrak{gl}(V) \,/\, k = \bar{k}$, $\mathrm{char}\, k = 0$.

if $\mathrm{tr}\, xy = 0$ $\forall x, y \in L$, then L is solvable.

Thm 1. (Cartan's Criterion) $L \,/\, k = \bar{k}$, $\mathrm{char}\, k = 0$. Then

$$L \text{ is solvable} \iff \mathrm{tr}(\underset{\in \mathfrak{gl}(L)}{\mathrm{ad}\,x} \circ \underset{\in \mathfrak{gl}(L)}{\mathrm{ad}\,y}) = 0 \quad \forall x \in L,\ y \in L' := [L, L].$$

Deduction of Thm 1 (generic) from Prop 1 (linear Lie algebra).

(Thm 1. L solvable $\Longleftrightarrow$ $\mathrm{tr}(\mathrm{ad}x\,\mathrm{ad}y) = 0$ $\forall x \in L, y \in L'$)

Pf (assuming Prop 1):

$(\Rightarrow)$. L solvable $\Longrightarrow$ $\mathrm{ad}L$ $\subseteq \mathfrak{gl}(L)$. is solvable $\Longrightarrow$ Since $\mathrm{ad}L \subseteq \mathfrak{gl}(L)$,

since it's the image of L under the ad-rep

$\mathrm{tr}(\mathrm{ad}x\,\mathrm{ad}y) = 0$ $\forall x \in L, y \in L'$.

(Ex: If $\tilde{L} \subseteq \mathfrak{gl}(V)$ and L is solvable, $\mathrm{tr}(xy) = 0$ $\forall x \in \tilde{L}, y \in \tilde{L}'$.)

diagonal entries are all 0.   upper Δ strictly upper Δ

$(\Leftarrow)$ $\mathrm{tr}(\mathrm{ad}x\,\mathrm{ad}y) = 0$ $\forall x \in \overset{?\,L'}{L}, y \in L'$

$\Longrightarrow$ By Prop 1, $\mathrm{ad}L'$ is solvable

$\Longrightarrow$ $\mathrm{ad}L' = L'/Z(L')$ and $Z(L')$ is solvable. So L' is solvable $\Rightarrow$ L is solvable. $\square$