Last time: We reduced

Engel's Thm: $L \subset \mathfrak{gl}(V)$, x nilp $\forall x \in L \Rightarrow \exists$ full flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$

s.t. $xV_i \subset V_{i-1}$ $\forall i$

(equiv, a Lie alg is nilp iff all its elts are ad-nilp.)

to

Lemma 1. $\exists$ $0 \neq v \in V$. s.t. $xv = 0$ under the same assumption.

Lie's Thm: $L \subset \mathfrak{gl}(V) \,/\, k = \bar{k}$, char $k = 0$, L solvable

$\Rightarrow \exists$ full flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ s.t. $xV_i \in V_i$ $\forall x \in L$

to

Lemma 2. $\exists$ $0 \neq v \in V$ s.t. $xv \in kv = \langle v \rangle$ $\forall x \in L$ under the same assumption.

Today. Proofs of the lemmas.

Proof of Lemma 1.

Preparation: observe that

(1) Given a nilp map $z \in \mathfrak{gl}(V)$, $V \neq 0$,

$\exists\ 0 \neq v \in V$ s.t. $zv = 0$: take any $0 \neq u \in V$. say $z^N = 0$,

then $z^N \cdot u = 0 \cdot u = 0$. Now take maximal k w/ $z^k u \neq 0$.

and take $v = z^k u$.

(2) Given a Lie algebra $L \subseteq \mathfrak{gl}(V)$ and $x \in L$,

If x is nilp, then x is ad-nilp.

Pf: Consider the maps $\lambda_x, \rho_x : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ w/ $\lambda_x(y) = xy$, $\rho_x(y) = yx$.

Then $\mathrm{ad}_x(y) = xy - yx = \lambda_x(y) - \rho_x(y) = (\lambda_x - \rho_x)(y)$

and $\lambda_x \rho_x = \rho_x \lambda_x$.

Say $x^N = 0$ for some N, then

$$(\operatorname{ad} x)^{2N+1} = (\lambda_x - \rho_x)^{2N+1} = \sum_{\substack{0 \le i,j \le 2N+1 \\ i+j = 2N+1}} \overset{\text{scalar}}{c_{i,j}} \underbrace{\lambda_x^i \rho_x^j}_{\max(i,j) > N} = 0 \qquad \square$$

(3) Given a Lie algebra $L \subseteq \mathfrak{gl}(V)$ and an ideal $A \subseteq L$, the subspace $V_{A,0} := \{ v \in V, \ av = 0 \ \ \forall a \in A \}$ is L-invariant.

Pf: Take $v \in V_{A,0}$, $y \in L$, then

$$a(yv) = (ya + [a,y])v = y\underbrace{av}_{=0} + \underbrace{[a,y]}_{\in A}v = 0 + 0 = 0 \quad \forall a \in A. \qquad \square$$

Pf of Lemma 1. ([EW] P.46) Induct on $\dim L$.

Base case: $\dim L = 1$. Then $L = \langle z \rangle$, z nilp by assump.

Done by observation (1).

Inductive Step: Strategy: We'll find a codimension-1 ideal $A \subseteq L$

so that $L = A \oplus \langle y \rangle$ for some $y \in L \setminus A$.

Then by induction, $V_{A,0} = \{v \in V,\ av = 0\ \forall a \in A\}$

is nontrivial. By ob. (3), $V_{A,0}$ is invariant under L

and hence invariant under y in particular. $V_{A,0} \circlearrowleft y$.

ob (1) implies that $\exists\ 0 \neq v \in V_{A,0} \subseteq V$ s.t. $yv = 0$.

It follows that $xv = 0 \quad \forall x \in L$.

Here's how we find the desired A:

- Take a maximal subalgebra $A \subseteq L$. Note $\exists$ a Lie alg rep

$$\varphi : A \longrightarrow \mathfrak{gl}(L/A)$$

$$a \longmapsto \left(\varphi(a) : L/A \to L/A, \quad x + A \longmapsto [a,x] + A \ \forall x \in L \right)$$

To be checked: ① $\varphi(a)$ is well-defined and linear $\forall a \in A$.

② φ is linear

$\Big\}$ routine.

③ φ respects brackets: $\forall a, b \in A \quad [\varphi(a), \varphi(b)] = \varphi[a,b]$

Ex: This follows from def. and the Jacobi identity.

- Consider the algebra

$$\phi(A) \subseteq \mathfrak{gl}(L/A).$$

Now, $\dim \varphi(A) \leq \dim A < \dim L$ and $\varphi(A) = \{\varphi(a) : a \in A\}$ consists

$\boxed{\varphi(\underline{\underline{A}}) \subseteq gl(\underline{\underline{L/A}})}$ entirely of nilp maps (a is nilp $\xRightarrow{\text{Ob}(2)}$ ad a nilp $\Rightarrow \varphi(a)$ nilp $\forall a \in A$)

So by induction. $\exists$ a coset $0 \neq y + A \in L/A$ s.t. $\varphi(a) \cdot (y + A) = 0 \ \forall a \in A$.

ie, $[a \cdot y] \in A \quad \forall a \in A$. Take $\tilde{A} = A \oplus \langle y \rangle$.

Then $\hat{A}$ is a subalgebra of L since $[a, y] \in A \ \forall a \in A$.

By maximality of A, $\tilde{A} = L$. But then it follows that

A is an ideal in L. This gives us the desired co-dim 1 ideal.

$\square$

Pf of Lemma 2. (Sketch)

The proof is similar to that of Lemma 1. We'll

- Induct on dim L. Base case: $\dim L = 1$, say $L = \langle z \rangle$, $z \neq 0$.

  $\exists\, 0 \neq v \in V$ s.t. $zv \in kV$ by JNF considerations.

  Jordan normal form

- To use induction, we'll find an ideal $A \subseteq L$ of codimension 1

  (Easy: L solvable $\Rightarrow$ $L/[L,L] \neq 0$ $\Longrightarrow$ can lift a codim-1 subalg/ideal in $L/[L,L]$ to a codim 1 ideal $A \subseteq L$ by the Correspondence thm)

  abelian, any subalg is an ideal

Now, by induction, $\exists\, 0 \neq v \in V$ s.t. $av \in kV$, say $av = \lambda(a)v$ $\forall a \in A$.

Think of λ as a function $\lambda : A \to k$, $a \longmapsto \lambda(a)$.

Then $V_{A,\lambda} := \{ v \in V : av = \lambda(a) v \quad \forall a \in A \}$

is non trivial. counterpart for $V_{A,0}$

Hard: $V_{A,\lambda}$ is invariant under L. (Invariance Lemma, EW. Lemma 5.5)

(uses char $k=0$)

The rest is the same as before:

A has codim 1 $\Rightarrow L = A \oplus \langle y \rangle$.

$V_{A,\lambda}$ is inv under L. $\Rightarrow V_{A,\lambda} \circlearrowleft y$.

Find e-vector $v \in V_{A,\lambda}$ of y. then $xv \in \langle v \rangle \quad \forall x \in L$. $\square$
(JNF)