Last time:

$\mathfrak{b}(n,k)$ [EW]

$\parallel$

- Def: derived series. solvable algebras. e.g. $\mathfrak{t}(n,k)$ upper Δ matrices

- Prop: (1) L solvable $\Rightarrow$ subquotients of L are solvable

  (2) $I \trianglelefteq L$ (ideal), L/I both solvable $\Rightarrow$ L solvable.

  (3) $I, J \trianglelefteq L$ solvable $\Rightarrow$ $I+J$ solvable $\rightsquigarrow$ Rad L (more on this later)

- Def: descending central series, nilpotent algebras, e.g. $\mathfrak{n}(n,k)$.
  strictly upper Δ mats.

- Prop: (1) L nilp $\Rightarrow$ subquotients of L nilp

  (2) $L/Z(L)$ nilp $\Rightarrow$ L is nilp.

  (3) L nilp $\Rightarrow$ $Z(L) \neq 0$.

Today. Engel's Thm and Lie's Thm. (char. of solvable and nilp algebras)

I. Engel's Thm

Def. An elt $x \in L$ (Lie algebra) is called ad-nilpotent if the map

$$\mathrm{ad}\,x \in \mathfrak{gl}(L) \text{ is nilpotent, i.e., if } (\mathrm{ad}\,x)^n = 0 \ \forall n >> 0.$$

observation: L is nilp $\Longrightarrow$ $\underbrace{[x[\cdots[x}_{(\mathrm{ad}x)^k}[x,[x,\underset{L}{\overset{\cap}{y}}]]]]] = 0 \ \forall y \in L \ \forall k >> 0.$ $\quad [\ldots] \in L^{k+1}$ $\quad \forall x \in L$

$\Longrightarrow$ $\mathrm{ad}\,x$ is nilp $\forall x \in L \Rightarrow x$ is ad-nilp $\forall x \in L$.

Engel's Thm: The converse is also true: if x is ad-nilp $\forall x \in L$,

then L is nilp.

The results: I. Nilpotency.

AI. Technical Lemmas (Hum Thm 3.3, EW Prop 6.2)

Suppose $L \subseteq gl(V)$ for a f.d.v.s. $V \neq 0$. If all elts of L are nilp, then $\exists\ 0 \neq v \in V$ s.t. $xv = 0\ \forall x \in L$. (simultaneous annihilation).

$\Downarrow$

(L is a subalgebra of $\mathfrak{n}(n,k)$)

BI: Engel's Thm. v.1. (Hum Cor 3.3, EW Thm 6.1)

$\Leftarrow$ Suppose $L \subseteq gl(V)$ for a f.d.v.s $V \neq 0$. If all elts of L are nilp, then V has a basis β w.r.t. which all elts of L are given by a strictly upper triangular matrix, ie, there is a (full) flag $0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ in V s.t. $x \cdot V_i \subseteq V_{i-1}\ \forall i$. (say $\beta = \{v_1, v_2, \ldots, v_n\}$. then take $V_i := \langle v_1, \ldots, v_i \rangle$)

$\Downarrow$

CI: Engel's Thm, v. 2: L is nilp $\Longleftrightarrow$ x is ad-nilp $\forall x \in L$.

II. Solvability.

A II Technical Lemma (Hum Thm 4.1, EW 6.6)

Let $L \subseteq \mathfrak{gl}(V)$ for a f.d.v.s $V \neq 0$ over an algebraically closed field k of characteristic 0. Then $\exists\, 0 \neq v \in V$ s.t. v is an eigenvector for x. $\forall x \in L$.

(simultaneous eigenvector)

$$\Downarrow$$

B II. Lie's Thm

Let $L \subseteq \mathfrak{gl}(V)$ for a f.d.v.s $V \neq 0$ over $k = \bar{k}$, char $k = 0$.

If L is solvable, then V has a basis w.r.t. which x is an upper triangular matrix, i.e., there is a flag $0 \subset V_1 \subset V_2 \cdots \subset V_n = V$ in V s.t. $x \cdot V_i \subseteq V_i$, $\forall x \in L$. ($\to L \subseteq \mathfrak{t}(n,k)$)

Pf of $BI \Rightarrow CI$, $AI \Rightarrow BI$, $AII \Rightarrow BII$.

Pf of "$BI \Rightarrow CI$":

flag version. ad-nilp char.

Now. $\text{ad}\, L \cong L/\ker \text{ad} = L/Z(L)$

Thus, $L/Z(L)$ is nilp.

So L is nilp. $\square$

Assume BI.

CI: L is nilp $\Leftrightarrow$ x is ad-nilp $\forall x \in L$.

$\Rightarrow$. follows from our earlier observations (using the def of central series. nilp)

$\Leftarrow$ Say x is ad-nilp $\forall x \in L$. Consider the map

$$\text{ad} : L \to \underset{\text{“}V\text{”}}{\mathfrak{gl}(L)} \overset{\beta}{\cong} \mathfrak{gl}(n,k)$$

and the image $\underline{\underline{\text{ad}\, L}} \subseteq \underset{\text{“}V\text{”}}{\mathfrak{gl}(L)}$. Since x is ad-nilp $\forall x \in L$, all elts in the image $\text{ad}\, L$ are nilp.

So by BI. there exists a basis β of L s.t. w.r.t. to it $\text{ad}\, L \subseteq \mathfrak{n}(n,k)$. So $\text{ad}\, L$ is nilp.

Pf of "AI ⇒ BI" : Let $L \subseteq \mathfrak{gl}(V)$ for a f.d. v.s $V \neq 0$.

simultaneous annihilation   flag version

We will prove the existence of the desired flag

$0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ by induction on $\dim V =: n$.

Base case: $n = 1$. Then $\exists\ 0 \neq v \in V$ s.t. $x v = 0$ $\forall x \in L$. so $V = \langle v \rangle$.

and we're done, $0 \subset V_1 = \langle v \rangle = V$.

Inductive Step: By AI, $\exists\ 0 \neq u \in V$. s.t. $xu = 0$ $\forall x \in L$. Let $U = \langle u \rangle$.

Then, $\forall x \in L$, the map $V \xrightarrow{x} V \xrightarrow{\text{proj}} V/U$, $v \mapsto x \cdot v + U$ induces a linear map

$$\bar{x} : V/U \longrightarrow V/U, \quad v \mapsto x \cdot v + U \quad \forall v \in V.$$

Now consider the map $\bar{\Phi} : L \longrightarrow \mathfrak{gl}(V/U)$, $x \mapsto \bar{x}$.

It's a Lie algebra hom (E.x.). $\dim V/U = \dim V - 1$

Now every elt $\bar{x} \in \operatorname{Im} \bar{\Phi}$ is nilp since x is nilp $\forall x \in L$. so we may apply A I on $\bar{\phi} : L \to gl(V/U)$.

Since $\dim(V/U) < \dim V$, by induction we have a basis

$\beta' = \{v_1, v_2, \dots, v_{n-1}\}$ of V/U w.r.t. which $\bar{x}$ is strictly

upper triangular $\forall x \in L$. ($x \cdot (v_i + U) \in \langle v_1, \dots, v_{i+1} \rangle + U \subseteq \underline{\langle v_1, \dots, v_{i+1}, u \rangle}$)
$\forall i$

It follows that w.r.t. the basis $\beta = \{u, v_1, v_2, \dots, v_{n-1}\}$,

then $\bar{x}$ is strictly upper triangular w.r.t. β $\forall x \in L$. $\square$

__Rmk:__ The proof of "A II $\Rightarrow$ B II" is entirely similar.

So, it remains to prove the two technical lemmas.