Last time: Finished classification of sl_2-irreps (f.d. / $\mathbb{C}$).

Today: Solvable and Nilpotent Lie algebras

Solvability.

Work over an arbitrary field k.

Def. (Derived series). The <u>derived series</u> of a Lie algebra L is the series

$$L^{(0)} := L \supset L^{(1)} := [L^{(0)}, L^{(0)}] \supset L^{(2)} := [L^{(1)}, L^{(1)}] \supset \cdots$$

where $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$.

Def. L is <u>solvable</u> if $L^{(n)} = 0$ for some n.

e.g. ① L abelian $\Rightarrow L^{(1)} = [L, L] = 0 \Rightarrow L$ solvable "abelian $\Rightarrow$ solvable".

② L simple $\Rightarrow L^{(1)} = \underset{\text{ideal}}{[L,L]} = L \Rightarrow L^{(i)} = L \neq 0 \ \forall i \Rightarrow L$ is not solvable

" simples $\Rightarrow$ not solvable "

Prop/E.g. $L = \underbrace{t(n,k)}_{\text{upper } \triangle \text{ matrices}}$ is a solvable Lie algebra.

Pf sketch: Repeatedly use $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$.

Note: (1). $L^{(1)} = [L,L] = n(n,k)$.

$t(n,k) = \underbrace{\delta(n,k)}_{\text{diagonals}} \oplus \underbrace{n(n,k)}_{\text{strictly upper } \triangle}$

$w \longrightarrow x + y$

$\longrightarrow$ $[w, w'] = [x+y, x'+y']$ $\downarrow$ only strictly upper $\triangle$ matrices survive

$L^{(i)} = 0 \ \forall i \gg 0$. $\square$

(2) Consider the 'level' of each matrix unit e_{ij}: $j-i$. $\longrightarrow L^{(i)} \in \operatorname{Span}\left(\text{matrix units of level } 2^{(i-1)}\right)$ $\forall i \geq 1$

Later: we'll see that all solvable Lie algebras are subalgebras of a Lie algebra iso to $t(n,k)$.

Prop: Let L be a Lie algebra.

(1) If L is solvable, then so are all subalgebras K of L and homomorphic images of L.

"quotient"

(2) If I is a solvable ideal of L and L/I is solvable, then L is solvable.

(3). If I, J are solvable ideals of L, then $I+J$ is solvable.

Pf: (1). "Subquotients".   ① If $K \subseteq L$ subalg., then clearly $K^{(i)} \subseteq L^{(i)}$ $\forall i$.

So $K^{(i)} = 0$ $\forall i >> 0$ since L is solvable.

② . Let $\phi: L \to L'$ be a hom. Note that $\phi(L^{(i)}) = (\phi(L))^{(i)}$

by induction (Ex). ......

(2). I, L/I are solvable $\Longrightarrow$ $(L/I)^{(n)} = L^{(n)}/I = 0$, i.e., $L^{(n)} \subseteq I$, for some n;

$I^{(m)} = 0$ for some m.

$\Longrightarrow$ $L^{(m+n)} = 0$.

quotient of solvable, so solvable (1)

(3) I, J solvable: $\quad I+J / J \cong I / I \cap J$ by the 2nd Iso Thm. $(0 + 0)$

solvable

$\Downarrow$ (2)

$I+J$ is solvable. □

Note: If S is a maximal solvable ideal, then for ideal I, the ideal $S+I$ is solvable by (3), so $S+I=S$ by maximality.

It follows that L must have a unique maximal ideal.

Def. We define the <u>radical</u> of L to be the unique maximal solvable ideal. We write it as $\mathrm{Rad}\, L$.

E.g. L is simple $\Longrightarrow$ $\mathrm{Rad}\, L = 0$

Def: We call L <u>semisimple</u> if $\mathrm{Rad}\, L = 0$.

$\Big\} \Longrightarrow$ "simples are semisimple."

E.g. For any L, $L/\mathrm{Rad}\, L$ is semisimple by the Correspondence Thm.

Nilpotency.

Def. The descending central series of L is the series

$$L^0 := L \supset [L, L^0] =: L^1 \supset L^2 := [L, L^1] \supset L^3 := [L, L^2] \supset \cdots$$

where $L^i = [L, L^{i-1}]$ $\forall i$.

Def. We say L is nilpotent if $L^n = 0$ for some n.

e.g. abelian algebras are nilpotent; nilpotent algebras are solvable.

e.g. Solvable algebras are not necessarily nilpotent.

e.g. $L = t(n,k) \Rightarrow L^1 = L^{(1)} = \eta(n,k) \Rightarrow L^2 = [L, L^1] = L^1 \Rightarrow L^i = \overset{\neq 0}{L^1}\ \forall i \geq 1.$

Prop: Let L be a Lie algebra. ($L \neq 0$)

(1) If L is nilp. then so is any subalgebra of L and any homorphic image of L.

(2). If $L/\underset{\text{center}}{\underline{Z(L)}}$ is nilp, then L is nilp

(3) If L is nilpotent, then $Z(L) \neq 0$.

Pf: Ex.