HW. $\mathfrak{g}$ = Lie algebra / k gens: x, y, z. rel: $\underline{[x\,y]=z}$, $[y\,z]=x$, $\underline{[z\,x]=y}$.

(a) $\mathfrak{g} \cong sl_2$ if $k = \mathbb{C}$.

$\downarrow$

better: find an explicit iso

$\phi : sl_2 \longrightarrow \mathfrak{g}$

$h \longmapsto$? (should be diag.)

taking $x = \phi(h)$ and $\phi(e), \phi(f)$

to be $\underline{\text{lin comb}}$ of y, z will work.

should have eigenvalue 2 and -2.

(b) $\mathfrak{g} \not\cong sl_2$ if $k = \mathbb{R}$.

$\downarrow$

say $\exists$ iso $\phi : sl_2 \longrightarrow \mathfrak{g}$.

Consider $\Phi : \phi \circ \operatorname{ad} h \circ \phi^{-1} \in \operatorname{End}(\mathfrak{g})$.

$\Phi = \operatorname{ad} \phi(h)$ [general fact]

$\downarrow$

should be diagonalizable.

$\downarrow$

say $\phi(h) = \alpha x + \beta y + \gamma z$.

$\downarrow$

can get matrix for $\operatorname{ad} \phi(h)$.

$\downarrow$

consider eigenvalues $\rightarrow$ contradiction ....

Last time: Classification of f.d. irreps of $sl_2(\mathbb{C})$.

$e \cdot V^{(i)} \subseteq V^{(i-1)}$, $V^{(i)} = \mathrm{Span}\{w, \dots f^i w\}$

V is a f.d. irr of $sl_2(\mathbb{C})$ $\Longrightarrow$ V has a highest wt vector, say w s.t. $h \cdot w = \lambda w$ and $e \cdot w = 0$ $\Longrightarrow$ for d max. $f^d \cdot w \neq 0$

h easy, $W = \mathrm{Span}\{\overset{\lambda}{w}, \overset{\lambda-2}{f w}, \dots, \overset{\lambda-2d}{f^d \cdot w}\} \subseteq V$

f should be inv. So $W = V$.

To show $V \cong V_d$, we'll (1) show $\lambda = d$ (2) find an iso $V \longrightarrow V_d$.

(1) Consider $tr(h\cdot) = tr(e \cdot f \cdot - f \cdot e \cdot) = 0$

w.r.t. $\{w, f \cdot w, \dots f^d \cdot w\}$, $\underline{tr(h\cdot) = \lambda + (\lambda - 2) + \dots (\lambda - 2d)}$

this is zero $\Rightarrow \lambda = d$ ✓

(2). Try $f^i \cdot w \mapsto X^{d-i} Y^i$ and adjust, since the h-eigenvalues should match.

Or, try $w \longmapsto X^d$ and then figure out the rest......

$$w \longmapsto X^d$$
$$f \downarrow \quad \circlearrowleft \quad \downarrow f$$
$$f \cdot w \longmapsto f \cdot X^d = d X^{d-1} Y$$

Ex: $S^d V \cong V_d$ (natural mod.)

(1) find an iso

(2) show $S^d \underline{V}$ is irr. $\langle e_1, e_2 \rangle$.

Ex: $V_3 \otimes V_3$ decomp. into irr. by 'Weyl's Thm'

but how?

Thm: For every positive int d, $\exists!$ irreducible module V of sl_2 w/ dim $d+1$. It is isomorphic to the module V^d of Ew. Ch.6. In particular, it has a basis $\{w, f \cdot w, \dots, f^d \cdot w\}$ where $h \cdot (f^i \cdot w) = (d - 2i) w \;\forall\, 0 \le i \le d$, $e \cdot w = 0$.