Last time:

- Chevalley involution for sl_2 $\quad \sigma: e \mapsto -f,\ f \mapsto -e,\ h \mapsto -h.$
- construction of a rep V_d of sl_2 of dim $d+1$, $d \in \mathbb{Z}_{\geq 0}$.

$$sl_2 \circlearrowright V_d = \begin{pmatrix} \text{deg } d \text{ part of} \\ \mathbb{C}[X, Y] \end{pmatrix}$$

basis: $X^i Y^j$, $i+j=d$

$$e = X\frac{\partial}{\partial Y}$$

$$f = Y\frac{\partial}{\partial X}$$

$$h = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$$

We checked this defines an sl_2-rep by checking that the actions respect the bracket relations.

Rmk: We can also check the relations by realizing the actions as matrices and checking the rels for the matrices.

e.g. $d=2$, $\beta = \{X^2, XY, Y^2\}$ $\quad [e]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $[f]_\beta \cdots$, $[h]_\beta \cdots$.

Today. Irr. of V_d. Classification of f.d. irreps of $\mathfrak{sl}_2$. (all V_d).

$V_d \cong S^d V$. (recall $\mathfrak{g} = \mathfrak{sl}_2 \circlearrowright V \longrightarrow \mathfrak{g} \circlearrowright V \otimes V$ Ex. $S^2 V = \dfrac{V \otimes V}{\langle v \otimes w - w \otimes v \rangle}$

$\rho(x) : V \otimes V \to V \otimes V$ $\xrightarrow{\text{descends}}$ $\rho(x) : S^2 V \to S^2 V$)

also works for n-fold symm. power.

Irr. of V_d.

There's a convenient visualization of the $\mathfrak{sl}_2$ actions on V_d (w.r.t. the basis $\{X^i Y^j\}_{i+j=d}$)

$$
\begin{tikzcd}
0 & Y^d \arrow[l] \arrow[r, bend left, "d"] \arrow[loop below, "-d"] & XY^{d-1} \arrow[l, bend left, "1"] \arrow[r, bend left, "d-1"] \arrow[loop below, "-d+2"] & \cdots \arrow[l, bend left, "2"] & X^{d-2}Y^2 \arrow[l, bend left] \arrow[r, bend left, "2"] \arrow[loop below, "d-4"] & X^{d-1}Y \arrow[l, bend left, "d-1"] \arrow[r, bend left, "1"] \arrow[loop below, "d-2"] & X^d \arrow[l, bend left, "d"] \arrow[r, "e"] \arrow[loop below, "d"] & 0
\end{tikzcd}
$$

e = blue

f = green

h = red.

$e = X \dfrac{\partial}{\partial Y}$ $\quad f = Y \dfrac{\partial}{\partial X}$ $\quad h = X \dfrac{\partial}{\partial X} - Y \dfrac{\partial}{\partial Y}$ (recall $h \cdot X^i Y^j = (i-j) X^i Y^j$.

Prop. V_d is irreducible $\forall\, d \geq 0$. Ex: (1) Prove this.

(2) Show that $V_d \cong S^d V$.

Classification of f.d. irreps of $\mathfrak{sl}_2(\mathbb{C})$.

Useful Observation : [EW. Lemma 8.3]. Say $\mathfrak{sl}_2 \circlearrowright V$ and $h \cdot v = \lambda \cdot v$ for some $v \in V$

Then (1) either $e \cdot v = 0$ or $e \cdot v$ is an e-vector of the h-action w/ e-value $\lambda + 2$.

(2) either $f \cdot v = 0$ or $f \cdot v$ is an e-vector of the h-action w/ e-value $\lambda - 2$.

$h \cdot e \cdot - e \cdot h \cdot = 2e \cdot$

Pf: (1). $h \cdot (e \cdot v) = e \cdot h \cdot v + 2e \cdot v = e \cdot \lambda v + 2 e \cdot v = (\lambda + 2) e \cdot v$.

(2). Similar.

Def: In a $\mathfrak{sl}_2$-rep V, a vector $v \in V$ is called a <u>highest weight</u> vector if it's an h-eigenvector and $e \cdot v = 0$.

Prop [EW Lemma 8.4] Existence of h.w.

Let V be a f.d. $\mathfrak{sl}_2$-rep. Then V contains a highest wt vector.

Pf (sketch): Since we work over $\mathbb{C}$, h has an eigenvector v, say with e-value λ: $h \cdot v = \lambda v$. Apply e's to v and 'use the Pigeonhole principle'. $\square$

Thm: [EW Thm 8.5]. Any f.d. irr. $\mathfrak{sl}_2$-rep V is iso to V_d for some $d \geq 0$.

Pf: (1). Take a h.w. vector $w \in V$. Let d be the maximal positive integer s.t. $f^d \cdot w \neq 0$.

Claim: $\gamma := \{w, f \cdot w, f^2 \cdot w, \ldots, f^d \cdot w\}$ is a basis for a submodule of V.

Pf of claim: ① linear independence: Consider eigenvalues.

② Invariance (i.e. why they span a submodule of V):

Use bracket relations in $\mathfrak{sl}_2$ to check

that $e \cdot v_i$, $f \cdot v_i$, $h \cdot v_i \in \gamma$ $\forall v_i := f^i \cdot w \in \operatorname{Span} \gamma$

interesting. easy easy

e.g. $e \cdot v_2 = e \cdot f \cdot (f \cdot w) = \underbrace{f \cdot \underbrace{e \cdot (f \cdot w)}_{\in V^{(1)}}}_{\in V^{(2)}} + \underbrace{h \cdot \underline{f \cdot w}}_{\in V^{(1)} \subseteq V^{(2)}}$

We can show that $\forall\ 0 \leq i \leq d$,

$$e \cdot v_i \in V^{(i)} := \operatorname{Span}\{v_0, v_1, \dots, v_{i-1}\} \subseteq \operatorname{Span} \gamma.$$

by induction on i.

① + ②. $W := \operatorname{Span} \gamma$ is a submodule of V. Since V is irr.

and $W \neq 0$, we have $V = W$.

Next time: show $\lambda = d$, then find iso $V_d \to V$.