Last time: Reps of Lie algebras $\phi: \mathfrak{g} \xrightarrow[\text{hom}]{\text{Lie alg}} \mathfrak{gl}(V)$. esp. the adj. rep.

Today: Chevalley involution ; reps of $\mathfrak{sl}_2$ (f.d. irr.)

The Chevalley involution, in two ways.

I. Note: · It makes sense to take the exponential of a nilpotent map ϕ.

$$\exp(\phi) = \sum_{n=0}^{\infty} \frac{\phi^n}{n!}$$

$\phi^N = 0$ for some N.

Example of a nilp. map: $\mathfrak{g} = \mathfrak{sl}_2$, adj rep. $\quad \mathrm{ad}\, e : \mathfrak{g} \to \mathfrak{g}$ is nilp.

Fact: (HW) The exp of a derivation δ of a Lie algebra $\mathfrak{g}$ (s.t. $\mathrm{ad}\, x$, $x \in \mathfrak{g}$) is an Lie algebra automorphism of $\mathfrak{g}$, e.g. $\exp \mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$ if $\mathrm{ad}\, x$ is nilp.

pf: (1) $\exp \delta$ is a hom : Leibniz rule $\quad$ (2). $\exp \delta$ has an inverse: find an explicit one.

Def: In $\mathfrak{sl}_2$, it now makes sense to define the map

$$\sigma = \exp \operatorname{ad} x \circ \exp \operatorname{ad}(-y) \circ \exp \operatorname{ad} x. \quad \left(\begin{array}{c} (x, y, h) \\ \parallel \\ (e, f, h) \end{array} \right).$$

Fact: The automorphism σ behaves nicely:

$$\sigma(x) = -y, \quad \sigma(y) = -x, \quad \sigma(h) = -h.$$

In particular, $\sigma^2 = \mathrm{id}$.

$$\phi \exp \operatorname{ad} x \, \phi^{-1} = \exp \operatorname{ad}(\phi(x))$$

We call σ the Chevalley involution.

Def: Given a Lie algebra $\mathfrak{g}$, an automorphism of $\mathfrak{g}$ of the form $\exp \operatorname{ad} x$ is called an inner automorphism.

Prop: (Ex) The set $\mathrm{Inn}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$ is a normal subgp of the gp $\mathrm{Aut}(\mathfrak{g})$ of all auto. of $\mathfrak{g}$.

II. Recall $\sigma = \exp \operatorname{ad} x \, \exp \operatorname{ad}(-y) \, \exp \operatorname{ad} x$.

Fact: (Ex). $(\exp z)(w)(\exp z)^{-1} = \exp(\operatorname{ad} z)(w)$.

Consequence: $\sigma(w) = \underbrace{sws^{-1}}_{\text{alternative way of realizing } \sigma \text{, namely, as a conj.}}$ where $s = \exp(x)\exp(-y)\exp(x)$

not every inv. map is of the form $\exp z$.

Fact: More generally, if $\mathfrak{g} < \mathfrak{gl}(V)$ and $s \in GL(V)$, then the map $\phi: \mathfrak{g} \to \mathfrak{g}$, $w \longmapsto sws^{-1}$ is a Lie algebra automorphism if $s\mathfrak{g}s^{-1} = \mathfrak{g}$.

Finite dimensional irreducible reps/modules of $sl_2(\mathbb{C})$.

<u>Omit:</u> def. of submodule, simple/irreducible module/rep, direct sum, indecomposable module, completely reducible, etc......

<u>Recall:</u> $sl_2(\mathbb{C}) = \langle e, f, h \rangle_{Lie} \Big/ \langle [e,f]=h,\ [h,e]=2e,\ [h,f]=-2f \rangle$,

So to define a rep $\phi: sl_2(\mathbb{C}) \longrightarrow gl(V)$, it suffices to

(1) assign linear maps $\phi(e), \phi(f), \phi(h) \in gl(V)$.

(2) extend the assignment to a unique linear map ($\{e,f,h\}$ is a basis of $sl_2(\mathbb{C})$)

(3) check that the brackets are respected, i.e., check the relations.

i.e., $[\phi(e), \phi(f)] = \phi(h)$, $[\phi(h), \phi(e)] = 2\phi(e)$, $[\phi(h), \phi(f)] = -2\phi(f)$

We'll use this strategy to def an irr. rep V_d of dim $d+1$ for every $d \in \mathbb{Z}_{\geq 0}$.

The construction of V_d. $\phi : \mathfrak{sl}_2 := \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathfrak{gl}(V_d)$.

The space V_d: the set of all degree-d polynomials in $\mathbb{C}[X, Y]$.

a basis we'll fix: $\{X^d, X^{d-1}Y, X^{d-2}Y^2, \ldots, X^1Y^{d-1}, Y^d\}$.

$\Downarrow$

$\dim V_d = d+1$.

The actions. $\phi(e) = X\dfrac{\partial}{\partial Y}$ $\quad \phi(f) = Y\dfrac{\partial}{\partial X}$

$$\phi(h) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$$

We need to check that this extends to a rep of $\mathfrak{sl}_2$. So we need to check the relations. We'll check each rel. on the basis.

e.g. $[\phi(e),\phi(f)] = \phi(h)$ ✓

$$X\frac{\partial}{\partial Y} \qquad Y\frac{\partial}{\partial X} \qquad X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Ex: Check the other two relations.

Ex: V_d is irreducible.

say $a,b \geq 1$ $\Bigg\{$

$$\left(X\frac{\partial}{\partial Y}Y\frac{\partial}{\partial X} - Y\frac{\partial}{\partial X}X\frac{\partial}{\partial Y}\right)\left(X^aY^b\right) = \left(X\frac{\partial}{\partial Y}\right)\left(aX^{a-1}Y^{b+1}\right) - \left(Y\frac{\partial}{\partial X}\right)\left(bX^{a+1}Y^{b-1}\right)$$

$$= (b+1)aX^aY^b - (a+1)bX^aY^b = (a-b)X^aY^b$$

$$\left(X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}\right)\left(X^aY^b\right) = (a-b)X^aY^b$$

$b=0$: similar

$a=0$. $\left(\cancel{X\frac{\partial}{\partial Y}Y\frac{\partial}{\partial X}} - Y\frac{\partial}{\partial X}X\frac{\partial}{\partial Y}\right)Y^d = -Y\frac{\partial}{\partial X}\cdot d\cdot XY^{d-1} = -Y^d\cdot d = \left(X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}\right)Y^b.$