Last time:

- Derivations of a "k-algebra" forms a Lie algebra.

- Basic notions: ideals, simple Lie algebras (e.g. $\mathfrak{sl}_2$),
  Iso Thms (hom.)

Today.

1. Representations

2. Chevalley Involution.

1. Representations of Lie Algebras.

Recall from ring theory that a representation of a (associative) k-algebra A is the data of a homomorphism $\rho: A \to \mathrm{End}(V)$, $a \mapsto \rho(a)$ where V is a vector space. equiv. it's the data of an A-module V s.t.

(1) a acts as a linear map on $\rho(a): V \to V$ $\forall a \in A$

(2) The map $\rho: A \to \mathrm{End}(V)$, $a \mapsto \rho(a)$ is linear.

(3) $\rho(a) \cdot \rho(b) = \rho(ab)$ $\forall a, b \in A$.

Def: A <u>representation of a</u> Lie algebra $\mathfrak{g}$ over k is a Lie algebra hom $\phi: \mathfrak{g} \to \mathfrak{gl}(V)$, equivalently it's the data of a $\mathfrak{g}$ <u>module</u> V s.t.

$\left(\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) \right)$

(1) a acts on V as a linear map $\phi(a) : V \to V$ $\forall a$.

(2) The map $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$, $a \longmapsto \phi(a)$ is linear.

(3) $\underset{\substack{\| \\ \phi(a)\phi(b) - \phi(b)\phi(a)}}{[\phi(a), \phi(b)]} = \phi([a,b])$ $\forall a, b \in \mathfrak{g}$.

Key Example: Every Lie algebra $\mathfrak{g}$ has an adjoint rep. defined by

$\phi : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ (so the module is $\mathfrak{g}$ itself)

$\phi(x)$ is the map in $\mathfrak{gl}(\mathfrak{g})$ s.t. $\phi(x)(y) = [x, y] =: \mathrm{ad}\,x(y)$

Check: (1) $\phi(x)$ is linear because $[\ ,\ ]$ is linear in the second coordinate.

(2) ϕ is linear, i.e., $\phi[(x_1 + x_2), (y)] = \phi[(x_1), (y)] + \phi[(x_2), (y)]$ This holds because $[\ ,\ ]$ is linear in the first coord.

$\phi[(cx_1), (y)] = c\,\phi[(x), (y)]$

13) Brackets are respected., i.e., $[\phi(x), \phi(y)]_{\mathfrak{gl}(\mathfrak{g})} = \phi([x,y]_{\mathfrak{g}})\ \forall x,y \in \mathfrak{g}$.

<u>i.e.</u> that $\phi(x)\phi(y) - \phi(y)\phi(x) = \phi([x,y]_{\mathfrak{g}})$

<u>i.e.</u> $(\operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x)(z) = \operatorname{ad}([x,y])(z)\ \forall z \in \mathfrak{g}$

<u>i.e.</u> $[x,[y,z]] - [y,[x,z]] = [[x,y],z]\ \forall z \in \mathfrak{g}.\ \forall x,y \in \mathfrak{g}$

This holds by the Jacobi identity.

<u>Rmk:</u> As always, having a rep of $\mathfrak{g}$ allows us to express elts of $\mathfrak{g}$ as matrices

e.g. $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\langle e,f,h\rangle / [e,f]=h,\ [h,e]=2e,\ [h,f]=-2f$.

So in the adj. rep, w.r.t to $\{e,f,h\}$ we have $\operatorname{ad} h = \begin{array}{c} \begin{matrix} e & f & h \end{matrix} \\ \begin{bmatrix} 2 & & \\ & -2 & \\ & & 0 \end{bmatrix} \end{array}$, $\operatorname{ad} e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

Eg. Given two $\mathfrak{g}$-modules V and W, the vector space $V \otimes W$ also has a $\mathfrak{g}$-module structure given by (Hw: we'll check this claim later using Hopf algebras)

$$g \cdot (v \otimes w) = (g \cdot v) \otimes w + v \otimes (g \cdot w).$$

(**Recall:** If V, W are reps of a gp G, then so is $V \otimes W$, w/ the action given by $g \cdot (v \otimes w) = gv \otimes gw \quad \forall g \in G, v \in V, w \in W.$)

Check the axioms: (1), (2): Ex. (3). $(g\cdot)(h\cdot) - (h\cdot)(g\cdot) = ([g,h]\cdot)$

It suffices to check (3) on pure tensors $v \otimes w$, $v \in V$, $w \in W$.

$$g \cdot h \cdot (v \otimes w) - h \cdot g \cdot (v \otimes w) = g \cdot (hv \otimes w + v \otimes hw) - h \cdot (gv \otimes w + v \otimes gw)$$

$$= \underline{ghv \otimes w} + \cancel{hv \otimes gw} + \cancel{gv \otimes hw} + \underline{\underline{v \otimes ghw}} - \underline{hgv \otimes w} - \cancel{gv \otimes hw} - \cancel{hv \otimes gw} - \underline{\underline{v \otimes hgw}}$$

$$= (ghv - hgv) \otimes w + v \otimes (gh - hg) w \overset{\text{because } V, W \text{ are reps}}{=\!=} [g,h] \cdot v \otimes w + v \otimes [g,h] w = [g,h](v \otimes w)$$

E.g. If $\mathfrak{g} \subset \mathfrak{gl}(V)$, then certainly $\mathfrak{g}$ acts on V.

e.g. $\mathfrak{g} = \mathfrak{sl}_2 \subset \mathfrak{gl}(\mathbb{C}^2)$, so $V = \mathbb{C}^2$ is a $\mathfrak{g}$-module.

$$e : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} : e_1 \mapsto 0, \ e_2 \mapsto e_1$$

$$f : \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} : e_1 \mapsto e_2 \quad e_2 \mapsto 0$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} : e_1 \mapsto e_1, \ e_2 \mapsto -e_2$$

$\mathfrak{g} = \mathfrak{sl}_2 \circlearrowright V = \mathbb{C}^2$

$$\begin{tikzcd} 0 & e_1 \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] \arrow[loop below, "h=1"] & e_2 \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] \arrow[loop below, "h=-1"] & 0 \end{tikzcd}$$

We'll call V the <u>natural module</u> of $\mathfrak{sl}_2$.

<u>Next time:</u> f.d. reps of $\mathfrak{sl}_2(\mathbb{C})$.

E.g. In the adjoint rep $\mathrm{ad}: \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, $x \mapsto \mathrm{ad}\, x$,

$$\ker \mathrm{ad} = \{x : \mathrm{ad}\, x = 0\} = \{x : [x, y] = 0 \ \forall y \in \mathfrak{g}\} = Z(\mathfrak{g})$$

Cor: $Z(\mathfrak{g})$ is an ideal in $\mathfrak{g}$.

Cor: If $\mathfrak{g}$ is simple, then $\ker \mathrm{ad} = 0$, so $\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is inj. So $\mathfrak{g}$ is linear. $\square$