Last time:

- Def of Lie alg hom / iso. Subalgebra

- Lie algebras (classical) of types $\underline{\underline{A}}$, $\underline{\underline{B, C, D}}$

includes

$sl_2 = \langle e, f, h \rangle$

$[h, e] = 2e$, $[h, f] = -2f$

$[e, f] = h$

come from bilinear forms

Today:

(1) Lie alg. of derivations of an algebra

(2) More basic notions: ideals, iso thms

(3) Representations of Lie algebras. The adjoint representation.

(1). Derivations.

setup: need a vector space $\mathcal{U}$ equipped w/ a bilinear product.

don't need associativity.

__Def:__ A derivation on/of $\mathcal{U}$ is a linear map on $\mathcal{U}$ s.t.

$$\delta(ab) = a\,\delta(b) + \delta(a)\,b \qquad (\text{'Leibniz rule'})$$

e.g. If $\mathfrak{g}$ is a Lie algebra, the map $\mathrm{ad}\,x = [x, -]$ is a derivation of $\mathfrak{g}$ (with $[\,,\,]$ as the bilinear prod on $\mathfrak{g}$).

__Prop:__ The set $\mathrm{Der}\,\mathcal{U}$ of derivations on $\mathcal{U}$ is a Lie subalgebra of $\underbrace{\mathfrak{gl}(\mathcal{U})}$.

$\mathrm{End}(\mathcal{U})$ under comm bracket.

__Need:__ check $\mathrm{Der}\,\mathcal{U}$ is closed under bracketing / taking commutators, i.e.,

HW. if $\delta_1, \delta_2 \in \mathrm{Der}\,\mathcal{U}$, then

$\forall a, b \in \mathcal{U}$.

$$[\delta_1\delta_2 - \delta_2\delta_1](ab) = \cdots\cdots = (\delta_1\delta_2 - \delta_1\delta_2)(a)\,b + a\,(\delta_1\delta_2 - \delta_2\delta_1)(b)$$

(2). More basic notions. (or, things you'd expect from experience in ring theory).

Let L be a Lie algebra.

(a) Def. An ideal in L is a subspace $I \subseteq L$ s.t. $[L, I] \subseteq I$, i.e., s.t. $[x, y] \in I$ $\forall x \in L, y \in I$.

$\left(\Downarrow \atop [y, x] \in I \right)$

Ideals in Lie algebras are analogous to normal subgps in gps and two-sided ideals in rings.

(b). Def. The center of L is the set $Z(L) = \{ z \in L, [x, z] = 0 \ \forall x \in L \}$

Prop: $Z(L)$ is an ideal in L.

Pf: $\forall x \in L, z \in Z(L)$ $\quad [y, [x, z]] = [y, 0] = 0 \quad \forall y \in L$.

(c). New ideals from old:

① If $I, J \subseteq L$ are ideals, then so is $I + J := \{ i+j : i \in I, j \in J \}$

② If $I, J \subseteq L$ are ideals, then so is $[I, J] = \{$lin comb of elts of the form $[i, j]$, $i \in I, j \in J\}$

eg. Def $[L, L]$ is called the derived algebra of L.

(d) Def: We say L is abelian if $[x, y] = 0 \quad \forall x, y \in L$.

So, L is abelian $\Longleftrightarrow [L, L] = 0 \Longleftrightarrow Z(L) = L$.

(e) Def: We say L is simple if L is not abelian and the only ideals of L are 0 and L itself.

Cor: If L is simple, then $Z(L)$ (an ideal) is 0.

Prop: $sl_2(k)$ is simple whenever $\operatorname{char} k \neq 2$.

Pf: Take an arbitrary elt $x = a\cdot e + b\cdot f + c\cdot h$ for some $a, b, c \in k$, $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(recall $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$)

We'll show that the ideal I containing x is the entire L.

$$[e, x] = a\cancel{[e,e]} + b[e,f] + c[e,h] = bh - 2ce.$$

$$[e, bh - 2ce] = [e, bh] = -2be$$

$$[f, -2be] = +2bh \qquad [f, 2bh] = -4bf$$

So, if $b \neq 0$, then $\{e, h, f\} \subseteq I$, so $I = L$.

Similarly, if $a \neq 0$, then $I = L$.

Finally, if $a = b = 0$ and $c \neq 0$, then $x = ch$ and $h \in I$.

Since $[e, h] = -2e$, $[f, h] = +2f$.

we have $e, f \in I$, so $I = L$.

(f). One can define the normalizer of a subalgebra of L.

and the centralizer of a subset of L.

and they'll turn out to be ideals of L. (HW)

(g). **Def:** Given an ideal $I \subseteq L$, the quotient v.s. L/I has the structure of a Lie algebra under the bracket given by

$[x + I, y + I] = [x, y] + I$ Ex: check well-definedness.

19) <u>Iso Thms</u>. Recall that a Lie algebra hom is a linear map $\phi : L_1 \to L_2$ s.t. $[\phi(x), \phi(y)] = \phi([x,y])$.

(1). If $\phi : L_1 \to L_2$ is a Lie algebra hom, then $L_1/\ker\phi \cong \mathrm{Im}\,\phi$ as Lie algebras.
$x + \ker\phi \mapsto \phi(x)$
*See the universal property of quotients in Humphreys.

(2). If I, J are ideals of L s.t. $I \subset J$, then J/I is an ideal of L/I, and $\frac{L/I}{J/I} \cong L/J$ (as Lie algebras).

(3). If I and J are ideals of L, then there is a natural iso between $I+J/J \cong I/I\cap J$ (as Lie algebras).

Rmk: The isomorphism are established by the usual linear maps from linear algebra. So to prove the thms it suffices to show that those maps respect brackets.