Last time:

- Def of Lie algebras
- Jacobi identity, derivation and Leibniz rule.

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$$

$$\Updownarrow$$

$$[x,[y,z]] = [[x,y],z]+[y,[x,z]]$$

- Examples of Lie algebras.
  - associative algeb as Lie algebra, $[\,,\,]$ = commutation
  - $\mathrm{End}_k(V) = \mathfrak{gl}(V) = \mathfrak{gl}(k,\underset{\dim V}{n}) \ (= \mathfrak{gl}_n(k))$
- Ado's Thm. If char $k=0$, then all Lie algebra are isomorphic to some subalgebra of a $\mathfrak{gl}_n(k)$.

Today. (A) Def of Lie algebra isomorphisms and subalgebras.

(B) More examples of Lie algebras

(C) Def of Lie algebra representations.

A. Def. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be Lie algebras. A Lie algebra homomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ is a linear map $\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$ s.t.

$$[\varphi(x), \varphi(y)] = \varphi([x, y]) \quad \forall x, y \in \mathfrak{g}_1.$$

A hom φ is an isomorphism if it's an isomorphism of v.s.

Def. A subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{g}' \subseteq \mathfrak{g}$ s.t. $[x, y] \in \mathfrak{g}'$ $\forall x, y \in \mathfrak{g}'$.

B. More examples of (linear) Lie algebras.

(a). The special linear algebra $sl_k(V)$

$$sl_k(V) = \{ \varphi \in gl(V) \mid tr(\varphi) = 0 \} = sl(n,k)$$

It's a subalgebra since $tr(\varphi_1\varphi_2 - \varphi_2\varphi_1) = tr(\varphi_1\varphi_2) - tr(\varphi_2\varphi_1) = 0$

Important special case: $sl_2 = \langle h, e, f \rangle$ a basis.

A_1

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Structure constants:

$$[h,e] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = 2e$$

$$\ldots \quad [h,f] = -2f \qquad [e,f] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = h$$

More generally, if $\dim V = n$, then $\dim \mathfrak{sl}(V) = n^2 - 1$.

You can easily find a linear basis for $\mathfrak{sl}(V)$.

The algebra $\mathfrak{sl}(V)$ is also known as the classical/linear Lie algebra of type A. $\mathfrak{sl}_k(\ell+1) \longrightarrow A_\ell$

(b c d) The linear/classical Lie algebras of types B, C, D.

Each of these algebras will arise from a bilinear form and will be a subalgebra of a special linear algebra

We'll compute the <u>symplectic algebra</u> C_n.

C_n. $\quad \mathfrak{g} \subseteq \mathfrak{gl}(V) \quad \dim V = 2n$

the bilinear form f on V is given by the Gram matrix

$$S_f = \left[\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right] \qquad S_{ij} = f(v_i, v_j)$$

$$\mathfrak{g} = \text{'}sp(V)\text{'} = \{ \varphi \in \mathfrak{gl}(V) \mid f(\varphi(v), w) = - f(v, \varphi(w)) \ \forall v, w \in V \}$$

Ex: - Check that $\mathfrak{g}$ is indeed a subalgebra of $\mathfrak{gl}(V)$

$\left(f((\varphi\phi - \phi\varphi)(v), w) = \ \ldots \ldots \ = -f(v, (\varphi\phi - \phi\varphi)(w)) \right)$

- check that $\mathfrak{g}$ wouldn't be a subalgebra of $\mathfrak{gl}(V)$ if the $-$ is missing.

Want: Describe the matrix realization of $sp(V)$ explicitly. find a basis and the dim.

Recall: $f(v,w) = v^T S_f w$.

$$Sp(V) = \{ \varphi \in gl(V) \mid f(\varphi(v), w) = -f(v, \varphi(w)) \ \forall v,w \in V \}$$

fix the basis

S uses

$$= \{ \varphi \in gl_{2n}(k) \mid [\varphi(v)]^T S w = -[v]^T S [\varphi(w)] \ \forall v,w \in V \}$$

$$= \{ \varphi \in gl_{2n}(k) \mid v^T [\varphi]^T S w = -v^T S [\varphi] w \ \forall v,w \in W \}.$$

$$= \{ \varphi \in gl_{2n}(k) \mid [\varphi]^T S = -S[\varphi] \}$$

Say $[\varphi] = \left[\begin{array}{c|c} m & n \\ \hline p & q \end{array}\right]$, then $[\varphi]^T = \begin{bmatrix} m^t & p^t \\ n^t & q^t \end{bmatrix}$

so

$$[\varphi]^T S = \begin{bmatrix} m^t & p^t \\ n^t & q^t \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} -p^t & m^t \\ -q^t & n^t \end{bmatrix}$$

$$-S[\varphi] = -\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} m & n \\ p & q \end{bmatrix} = -\begin{bmatrix} p & q \\ -m & -n \end{bmatrix}$$

So $\quad Sp(V) \overset{*}{=} \left\{ [\varphi] = \begin{bmatrix} m & n \\ p & q \end{bmatrix} \;\middle|\; p = p^t,\ n = n^t,\ \underline{\underline{m^t = -q}} \right\}$

$\Downarrow$

$tr(\varphi) = 0$

so $\quad sp(V) \subseteq sl(V)$

Ex: Use $(*)$ to find a basis for $sp(V)$, then deduce $\dim sp(V)$.

$\searrow$ $2\ell^2 + \ell$.

Rmk: We could have given f via a different (conj) Gram matrix,

e.g. $\left[\begin{array}{c|c} 0 & \begin{smallmatrix} 0 & & 1 \\ & \cdot & \\ 1 & & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} & & -1 \\ & \cdot & \\ -1 & & 0 \end{smallmatrix} & 0 \end{array}\right]$. This will lead to a different matrix realization.

The bilinear form for types B and D are given by

$$B: \begin{array}{cl} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{bmatrix} & \begin{array}{l} \text{— one row} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{blocks} \end{array} \\ \text{one col} & \end{array} \qquad D: \begin{bmatrix} 0 & I_\ell \\ I_\ell & 0 \end{bmatrix}$$

More linear Lie algebras (via matrices).

$$\mathfrak{t}(n,k) = \{\text{upper triangular } n \times n \text{ matrices}\}$$

$$\mathfrak{n}(n,k) = \{\text{strictly upper } \Delta \ n \times n \text{ matrices}\} - \begin{bmatrix} 0 & & * \\ & 0 & \\ 0 & & \ddots \ 0 \end{bmatrix}$$

etc.

Rmk: When dealing with matrices, we'll frequently use the simple fact that

$$e_{ij} e_{kl} = \delta_{jk} e_{il}.$$