Topics in Algebra: Introduction to Lie Algebras and Representation theory.

Time: 9:00 – 9:50 MWF

Place: Zoom / Microsoft Teams

Office Hours: Mondays 3 – 5 pm + appointments

Website: math.colorado.edu/~tixu6187/lie.html

HW: Collected every second Friday. Posted on website, updated after every lecture.

Textbook: - Introduction to Lie Algebras and Representation Theory. Main text

by Humphreys

- Introduction to Lie Algebras

by Erdmann & Wildon.

Ch 1. Basic Concepts.

- Def of Lie algebras.

Def. A Lie algebra over a field k is a k-vector space L with a bracket (binary) operation $[\,,\,] : L \times L \to L$ s.t.

(1) The bracket is bilinear.

(2) $[x,x]=0 \quad \forall x \in L$ (as usual, this implies $[x,y]=-[y,x]$ $\forall x,y \in L$ if $\operatorname{char} k \neq 2$)

consider $[x+y,x+y]=0$.

(3) $[x[yz]]+[y,[z,x]]+[z,[x,y]]=0$ Jacobi identity.

Rmk: rewrites to: $[x,[y,z]] = -[y,[z,x]]-[z,[x,y]]$

$$= [[z,x],y]+[[x,y],z]$$

$$= [y,[x,z]]+[[x,y],z]$$

So the map $\mathrm{ad}_x : w \mapsto [x,w]$ is a derivation in the sense that $\mathrm{ad}_x [y,z] = [\mathrm{ad}_x(y),z]+[y,\mathrm{ad}_x(z)]$

More generally, a derivation on a Lie algebra is a map d

Such that $d[y, z] = [d(y), z] + [y, d(z)]$

Leibniz rule

(think $(fg)' = f'g + fg'$ from calculus).

" Jacobi $\Longleftrightarrow$ $ad_x = w \mapsto [x, w]$ is a derivation $\forall x$ ".

Examples of Lie algebras.

(a). Let A be any associative algebra. Then A has the structure of a Lie algebra if we define $[x, y] = \underline{xy - yx}$.

'commutator'

Ex. Check the axioms: (1), (2) ✓ (3).

3) Check $\mathrm{ad}_x : w \longmapsto xw - wx$ is a derivation. $\forall x \in A$.

$$\mathrm{ad}_x([y,z]) = \mathrm{ad}_x(yz - zy) = x(yz - zy) - (yz - zy)x$$

$$= \ldots \qquad \Rightarrow \ldots$$

$$[\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)] = [xy - yx, z] + [y, xz - zx]$$

Rmk: As a Lie algebra, $[\,,\,]$ (commutation) is not necessarily associative:

$$[[x,y],z] \quad \text{vs} \quad [x,[y,z]]$$

$$(xy - yx)z - z(xy - yx) \qquad x(yz - zy) - (yz - zy)x$$

$$= xyz - yxz - zxy + zyx \qquad = xyz - xzy - yzx + zyx$$

doesn't appear here

(b). Given any v.s. V, the space $A = \mathrm{End}_k(V)$ of endomorphisms of V $\exists$ an associative algebra w/ composition as multip. so we can make it a Lie algebra with bracket

$$[f, g] = fg - gf$$

This Lie algebra is denoted by $\mathfrak{gl}(V)$ and called the a general linear (Lie) algebra.

If we pick a basis for V, then $\mathfrak{gl}(V) \cong \mathfrak{gl}(n, k)$ where $n = \dim V$.

<u>Def</u>: A Lie algebra L is called a <u>linear Lie algebra</u> if it's <u>isomorphic</u>. to a <u>subalgebra</u> of an algebra of the form $\mathfrak{gl}(V)$.

Ado's Thm: Over a field of char 0, every fin. dim. Lie algebra $\supset$ linear.