Math 2001. Lecture 9.

06. 13. 2022.

Last time:

- The binomial thm

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

- Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \cdots & & & & \end{array}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \forall\, 0<k<n.$$

$$\Updownarrow$$

$$\begin{array}{ccc} \binom{n-1}{k-1} & & \binom{n-1}{k} \\ & \binom{n}{k} & \end{array}$$

Today:

- a bit more on binomial expansions
- the inclusion-exclusion principle
- multisets. the bars-and-stars method.

1. Binomial expansions

Example: Use the binomial thm to find the coeff. of x^8 in $(x+2)^{13}$.

Soln:
$$(x+\underset{y}{\underline{2}})^{13} = \sum_{i=0}^{13} \binom{13}{i} x^i 2^{13-i}$$

The power x^8 occur only in the summand where $i=8$. i.e. only in the term $\binom{13}{8} x^8 \cdot 2^{13-8} = \binom{13}{5} \cdot 2^5 \cdot x^8$, so the desired coefficient is $\underline{\binom{13}{5} \cdot 2^5}$.

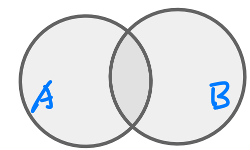
E.g. Similarly, the coefficient of $x^6 y^3$ in $(3x-2y)^9$ is

. . . .
$$\binom{9}{6}(3x)^6(-2y)^3,$$

so it's $\binom{9}{3} \cdot 3^6 \cdot (-2)^3 = -2^3 \cdot 3^6 \cdot \binom{9}{3}$

2. The inclusion-exclusion principle $\rightarrow$ for counting unions of sets

Prop 1. If A and B are finite sets, then $|A \cup B| \overset{*}{=} |A| + |B| - |A \cap B|$.



Pf: To get $|A \cup B|$ we need to each elt in A or B exactly once. The expression $|A| + |B|$ counts every elt in $A \setminus B$ or $B \setminus A$ once but counts every elt in $A \cap B$ twice.

It follows that $|A| + |B| - |A \cap B|$ counts $|A \cup B|$.

Rmk: When $A \cap B = \emptyset$, we have $|A \cap B| = 0$ so $(*)$ says $|A \cup B| = |A| + |B|$, recovering the addition principle.

(In other words, Prop 1 generalizes the addition principle.

Examples:

· (3.17) A 3-card hand is dealt off a usual 52-card deck. How many such hands are there that are all red or all faces (J, Q, K)?

Soln: Let A be the set of 3-card hands that are all red.

-- B -- . - -- - . - - -- all faces.

Then we need $|A \cup B|$, which equals

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= \binom{26}{3} + \binom{3\times 4}{3} - \binom{3\times 2}{3}$$

$$= \binom{26}{3} + \binom{12}{3} - \binom{6}{3}.$$

· (3.7.3) How many 4-digit positive numbers are there that are even or contain no zeros?

recall: an int. is even iff its last digit is even.

Answer:

$$\left|\underset{\text{even}}{\text{"}A\text{"}}\right| + \left|\underset{\text{no zero}}{\text{"}B\text{"}}\right| - |A \cap B|.$$

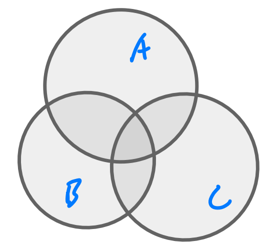
$$= 9 \cdot 10 \cdot 10 \cdot 5 + 9 \cdot 9 \cdot 9 \cdot 9 - 9 \cdot 9 \cdot 9 \cdot 4.$$

Prop 1. generalizes:

Prop 2: If A, B, C are three finite sets, then

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|.$$

Rmk: We can again prove the prop. by checking that that every elt in every region in the Venn diagram gets counted exactly once eventually on the RHS.



(representative examples):

$x\in A\setminus(B\cup C)$: $1+0+0-0-0-0+0=1$

$x\in (B\cap C)\setminus(A\cap B\cap C)$: $0+1+1-0-0-1+0=1$

$x\in A\cap B\cap C$: $1+1+1-1-1-1+1=1.$

The full generalization of Prop 2 is :

Thm. (The inclusion-exclusion principle) If $A_1, A_2, \ldots, A_n$ are n finite sets, then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{\substack{Y \subseteq X \\ |Y|=k}} \left| \bigcap_{A \in Y} A \right| \right)$$ where $X = \{A_1, \ldots, A_n\}$.

E.X. Prove the theorem.

(Hint: Let $a \in \bigcup_{i=1}^{n} A_i$. Suppose a is contained in exactly k of the n sets $A_1, \ldots, A_n$, $\rightarrow$ check that a is counted exactly once on the RHS.)

3. Multiset combinations, the bars-and-stars method.

A <u>multiset</u> is a variation of a set where each elt can appear multiple times. The number of times an elt appears is called its <u>multiplicity</u>. We enclose multisets in [,].

<u>e.g.</u> As sets, $\{1, 2, 2, 3, 3, 3\} = \{1, 2, 3\} = \{3, 1, 2\}$.

As multisets, $[1, 2, 2, 3, 3, 3] = [3, 1, 2, 2, 3, 3]$

but $[1, 2, 2, 3, 3, 3] \neq [1, 2, 3]$

$[1, 2, 2, 3, 3, 3] \neq [1, 2, 3, 3, 3]$.

Q: Given a set X of size n, how many multisets of size k are there whose elts are from X?

E.g. How many multisets of size 3 can be formed from the elts in the set $X = \{a, b, c, d\}$?

The numbers are small. Let's try listing all possibilities. (write xyz for $[x, y, z]$)

1 letter: aaa, bbb, ccc, ddd.

2 letters: aab, abb; aac, acc; aad, add;

bbc, bcc; bbd, bdd; ccd, cdd;

3 letters: abc, abd, acd, bcd $\rightarrow \binom{4}{3}$

Total: $4 + 12 + 4 = 20$.

A better organization / encoding scheme :

"use three bars to separate the different letters"

\# bars = n-1

e.g. a b b $\longleftrightarrow$ * | ** | |

\# a's, # b's, # c's, # d's should add to the desired size for the multisets, i.e. # stars = k.

b c d $\longleftrightarrow$ 0, 1, 1, 1

| * | * | *

e.g. a b d $\longleftrightarrow$ * | * | | *

d d d $\longleftrightarrow$ | | | ***

Point: There is a bijection between

$$\left\{ \begin{array}{l} \text{the multisets of size } k \\ \text{with elts from a set of } n \text{ things} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{bars-and-stars configurations,} \\ \text{with } k \text{ stars and } (n-1) \text{ bars} \end{array} \right\}.$$

The number of the latter type of configurations is

$$\binom{(n-1)+k}{k} = \binom{(n-1)+k}{n-1}.$$

For our problem, the desired number of multisets ($k=3$, $n=4$)

$$= \binom{(4-1)+3}{3} = \binom{6}{3} = 20.$$

More examples:

- \# multisets of size 5 formed using $\{a, b, c, d\}$ $= \binom{(4-1)+5}{5}$.

  e.g. $a|b|cc|d \leftrightarrow *|*|**|*$

- A bag has 20 (identical) red marbles, 20 green marbles, and 20 blue marbles. You reach into the bag and grab 20 marbles. How many outcomes are there?

  $\Updownarrow$

  \# multisets of size 20 formed using $\{R, G, B\}$ $= \binom{(3-1)+20}{20} = \binom{22}{2}$

  $= 231$.

- How many nonnegative integer solns are there for the equation

$$w+x+y+z = 20 \quad ?$$

(E.g. $(w,x,y,z) = (2,3,5,10) \longleftrightarrow **|***|*****|\underbrace{*\cdots*}_{10}$.

Does $|||\underbrace{**\cdots*}_{20}$ make sense (correspond to a soln?

Yes, it corresponds to the soln $(x,y,z,w) = (0,0,0,20)$

thanks to the fact that w,x,y,z can be zero.)

Answer: $\binom{20+(4-1)}{4-1} = \binom{23}{3}$.

- What about the number of positive integer soln to

$$x + y + z + w = 20 \ ?$$

Note: $|||\underbrace{* \cdots *}_{20}$ no longer corresponds to a legitimate soln !

Soln: Under the (change-of-variable) correspondence

$$w \longmapsto w' := w-1, \quad x \mapsto x' := x-1, \quad y \mapsto y' := y-1,$$

$z \mapsto z' := z-1$, the positive solns (x, y, z, w) of $x+y+z+w=20$ corresponds bijectively to the nonnegative solns (x', y', z', w') of

$$(x'+1) + (y'+1) + (z'+1) + (w'+1) = 20.$$

$$(2, 3, 5, 10) \longleftrightarrow (1, 2, 4, 9)$$

i.e.,

$$x' + y' + z' + w' = 16,$$

So the desired number is $\binom{16+(4-1)}{4-1} = \binom{19}{3}$.