Math 2001. Lecture 8.

Last time:

- Combinatorial proofs.

$$\binom{n}{k} = \binom{n}{n-k}, \quad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \quad 2^n = \sum_{i=0}^{n} \binom{n}{i}$$

L.6.

- Counting worksheet

Today:

- The binomial thm and Pascal's Triangle

another comb. proof

- Midterm.

1. The binomial thm and Pascal's Triangle

Thm: $\forall n \in \mathbb{Z}_{\geq 1}$, we have $$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \quad \left(= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i \right)$$

Eg.

$$(x+y)^1 = 1 \cdot x + 1 \cdot y$$

$$(x+y)^2 = 1 \cdot x^2 + 2xy + 1 \cdot y^2$$

$$\binom{3}{1}$$

$$(x+y)^3 = 1 \cdot x^3 + 3x^2y + 3xy^2 + 1 \cdot y^3$$

$$\binom{4}{2}$$

$$(x+y)^4 = 1 \cdot x^4 + 4 \cdot x^3 y + 6 \cdot x^2 y^2 + 4 \cdot xy^3 + 1 \cdot y^4$$

. . . .

$$\binom{8}{2}$$

Eg. $$(x+y)^8 = \sum_{i=0}^{8} \binom{8}{i} x^i y^{8-i} = x^8 + 8 \cdot x^7 y + 28 \cdot x^6 y^2 + 56 x^5 y^3 + 70 \cdot x^4 y^4 + 56 x^3 y^5 + \cdots$$

Pf: One way to expand $(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ copies}}$ is to sum all the 2^n summands of the form $x^i y^{n-i}$ which each results from multiplying n terms, with one from each pair of parentheses and i of them being x.

(e.g. $(x+y)^2 = (x+y)(x+y) = x\cdot x + x\cdot y + y\cdot x + y\cdot y = x\cdot x + 2xy + y\cdot y$.

$(x+y)(x+y)(x+y) = x\cdot x\cdot x + x\cdot x\cdot y + x\cdot y\cdot x + x\cdot y\cdot y + y\cdot x\cdot x + y\cdot x\cdot y + y y x + y\cdot y\cdot y$.

$= \cdots\cdots + 3\cdot x^2 y + \cdots$)

It follows that the coefficient of $x^i y^{n-i}$ in the expansion of $(x+y)^n$ is $\binom{n}{i}$, corresponding to the possibilities of picking i pairs from n pairs. $\square$

Rmks: (1) We can prove the result $2^n = \sum_{i=0}^{n} \binom{n}{i}$ from yesterday by using the binomial thm now:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \xrightarrow{\text{set } x=1=y} (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot 1^i \cdot 1^{n-i},$$

i.e. $2^n = \sum_{i=0}^{n} \binom{n}{i}$.

(2) Similarly. If $n=7$, $x=1$, $y=-1$, we recover from the binomial theorem

That equation from L6. and this "generalizes beyond the case $n=7$" to all $n \in \mathbb{Z}_{\geq 1}$.

$$(1-1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} (-1)^i = \binom{n}{0} \cdot 1 \cdot \underbrace{1}_{(-1)^0} + \binom{n}{1} \cdot 1 \cdot \underbrace{(-1)}_{(-1)^1} + \binom{n}{2} \cdot 1 \cdot 1 + \cdots\cdots$$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots\cdots$$

$n=7$: $0 = \binom{7}{0} - \binom{7}{1} + \binom{7}{2} - \binom{7}{3} + \binom{7}{4} - \binom{7}{5} + \binom{7}{6} - \binom{7}{7}$

$$\Rightarrow \binom{7}{1} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} = \binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6}.$$

Pascal's Triangle:

In light of the binomial thm, the numbers of the form $\binom{n}{i}$ $\left(= C(n,i)\right)$ are also often called <u>binomial coefficients</u>.

Binomial coefficients can be arranged into the so-called <u>Pascal's Triangle</u>.

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \cdots\cdots & & & &
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 \, \binom{3}{1} & & 3 \, \binom{3}{2} & & 1 & \\
1 & & 4 & & 6 \, \binom{4}{2} & & 4 & & 1 \\
 & & & 10 & \cdots\cdots & & & &
\end{array}
$$

Note:

- Each interior elt in a row is the sum of the numbers at its two shoulders, because $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ whenever $0 < k < n$.

  right shoulder   left shoulder

- This allows us to compute Pascal's Triangle inductively (recursively) by doing addition, without computing bin. coeff. as quotients.