Math 2001. Lecture 7.

06.09.2022.

Last time:

- permutations vs. combinations (subsets) : def, notation, computations
- more counting problems
- combinatorial identities and proofs, e.g. "$C(n,k) = C(n,n-k)$"
  $$\binom{n}{k} = \binom{n}{n-k}$$
  for int. n, k with $n \geq k \geq 0$.

Today:

- more comb. identities and proofs, including the binomial theorem.
- Counting worksheet

1. Combinatorial identities

Example 1. Yesterday we saw that

$$\underbrace{\binom{7}{1}+\binom{7}{3}+\binom{7}{5}+\binom{7}{7}}_{\#\text{ 7-digit binary strings with an odd number of 1s}} \overset{\overset{\overset{128 \,\to\, \#\text{ all 7-digit binary strings}}{2}}{\overset{\shortparallel}{64}}}{\overset{\shortparallel}{=}} \underbrace{\binom{7}{0}+\binom{7}{2}+\binom{7}{4}+\binom{7}{6}}_{\#\text{ 7-digit binary strings with an even number of 1s}}$$

Pf 1: We could match the summand on the two sides: since $\binom{n}{k}=\binom{n}{n-k}$,

we have $\binom{7}{1}=\binom{7}{6}$, $\binom{7}{3}=\binom{7}{4}$, $\binom{7}{5}=\binom{7}{2}$, $\binom{7}{7}=\binom{7}{0}$.

Pf 2 (combinatorial) LHS = # 7-digit bin. strings with an odd number of 1s.

|| by "symmetry", the distinction between the symbols 0 and 1 is superficial

= # 7 .. - - - - - - - - - - - - (odd) - - - of 0s.

|| same condition since 7 is odd

= # 7-digit . . . . even number of 1s.

= RHS.

Example 2: Prove that for all integers k, n with $0 < k < n$, we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

e.g. $\binom{7}{6} = \binom{6}{6} + \binom{6}{5}$, $\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$

$7 \quad 1 \quad 6$ ✓ $\qquad 10 \quad 4 \quad 6$ ✓

Pf 1: (algebraic)

$$\text{RHS} = \frac{(n-1)!}{k!\,(n-1-k)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$= \frac{(n-1)!\,(n-k)}{k!\,(n-1-k)!\,(n-k)} + \frac{k\,(n-1)!}{k\,(k-1)!\,(n-k)!}$$

$$= \frac{(n-1)!\,(n-k) + k\,(n-1)!}{k!\,(n-k)!}$$

$$= \frac{(n-1)!\,[n-k+k]}{k!\,(n-k)!} = \frac{(n-1)!\,n}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k} = \text{LHS}.$$

Pf 2: (combinatorial) Consider the task T of picking k elts out of the set $A = \{a_1, a_2, \cdots, a_n\}$ with n elts.

On the one hand, by the definition of $\binom{n}{k}$, we know that T can be performed in $\binom{n}{k}$ ways.

On the other hand, the picked elts are either all k elts from $\{a_1, a_2, \cdots, a_{n-1}\}$, (i) or they include a_n and $(k-1)$ elts from $\{a_1, \cdots, a_{n-1}\}$. These possibilities correspond to $\underbrace{\binom{n-1}{k}^{(ii)}}_{(i)}$ and $\underbrace{\binom{n-1}{k-1}}_{(ii)}$ configurations, respectively.

It follows that T can be performed in $\binom{n-1}{k} + \binom{n-1}{k-1}$ ways.

It further follows that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. $\square$

Example 3: For every $n \in \mathbb{Z}_{\geq 0}$, we have

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}. \rightarrow P(n)$$

e.g. $n=0$. $LHS = 2^0 = 1$, $RHS = \binom{0}{0} = 1$

$n=1$. $LHS = 2^1 = 2$, $RHS = \binom{1}{0} + \binom{1}{1} = 1+1 = 2$

$n=2$. $LHS = 2^2 = 4$, $RHS = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1+2+1 = 4$

$n=3$. $LHS = 2^3 = 8$, $RHS = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1+3+3+1 = 8$. ✓

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$n=6$ $LHS = 2^6 = 64$. $RHS = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$

$= 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64.$

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Pf 1 : (algebraic) One way : mimic and generalize the following. ← "mathematical induction"

Suppose we have proven $P(1)$, $P(2)$. $P(3)$, $P(4)$; we prove $P(5)$.

We will use the help of Example 2. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

reduce "n choose ..." to "(n-1) choose ..."s.

$$\text{RHS for } P(5) = \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$$

$$= \binom{4}{0} + \left[\binom{4}{1} + \binom{4}{0}\right] + \left[\binom{4}{2} + \binom{4}{1}\right] + \left[\binom{4}{3} + \binom{4}{2}\right] + \left[\binom{4}{4} + \binom{4}{3}\right] + \binom{4}{4}$$

$$= 2 \cdot \left[\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}\right]$$

$$= 2 \cdot 2^4 \quad \text{by } P(4)$$

$$= 2^5. \quad \to \text{So } P(5) \text{ holds.}$$

Pf 2: (combinatorial) Consider the task of counting all subsets of a set X with n elts, i.e., counting the power set $\mathcal{P}(X)$.

Recall that $|\mathcal{P}(X)| = 2^n$. On the other hand, a subset of X can have $0, 1, 2, \ldots$ or n elts (and there's no other possibility). The number of such sets are exactly $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}$. It follows that

$$2^n = |\mathcal{P}(X)| = \binom{n}{0} + \cdots + \binom{n}{n},$$ so we are done. $\square$

Pf 3: algebraically derive the fact $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ from the so-called binomial theorem. $\rightarrow$ tomorrow.