Math 2001. Lecture 6. 06.08.2022.

Last time:

- The mult., addition, and subtraction principles for counting
- Counting problems (ordered lists)

Today:

- more counting problems : permutations vs. Combinations (unordered)
- The binomial theorem.

Warm-up problems:

1. A password must be five characters long, made from the English alphabet, and have at least one upper-case letter.

(a) How many different such passwords are there? $\rightarrow 52^5 - \underbrace{26^5}_{\text{all lower case}}$

(b) What if we want a mix of upper and lower cases? $\rightarrow 52^5 - 26^5 \cdot 2.$

2. Line up 5 cards out of a standard 52-card deck. How many such line-ups have only reds or only clubs? (black)

$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 + 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 .$$

1. Permutations, factorials, and $P(n,k)$

Def: For each $n \in \mathbb{Z}_{\geq 1}$, we define the factorial of n to be the number

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 .$$

We also define $0! = 1$.

e.g. $0! = 1$, $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$,

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 3! = 4 \cdot 6 = 24$, $5! = 5 \cdot 24 = 120$, ...

Def. (Permutations / k-permutations) Let X be a set with n elements. Let k be an integer with $0 \le k \le n$. A <u>k-permutation of X</u> is a non-repetitive list of k-elts selected from X.

if $k = n$, we also just call a k-permutation ($= n$-permutation) of X a <u>permutation</u> of X.

Thus (by yesterday's lecture): if $|X| = n$, then

$$\# \text{ } k\text{-permutations of } X = \underbrace{\underset{n-0}{n} \cdot (n-1) \cdot \cdots \cdot (n-k+1)}_{k \text{ factors descending from } n.} ;$$

In particular, if $k = n$, then

$$\# \text{ } k\text{-perm. of } X = \# \text{ perm of } X = n \cdot (n-1) \cdots \cdot 1 = n!$$

Rmks on notation and computation :

$\#$ k-perm. of n objects

$\|$

- We often denote the product $n \cdot (n-1) \cdot \cdots (n-k+1)$ by $P(n,k)$.

e.g. $P(6,2) = 6 \cdot 5$

- We note that $P(n,k) = n \cdot (n-1)(n-2) \cdots (n-k+1)$

$$= \frac{n(n-1)\cdots(n-k+1) \cdot [(n-k)(n-k-1)\cdots 1]}{[(n-k)(n-k-1)\cdots 1]}$$

$$= \frac{n!}{(n-k)!}$$ $\to$ a more compact expression.

(not that practical / efficient for computation, though)

e.g. $P(6,2) = 6 \cdot 5 = \frac{6 \cdot 5 \cdot [4 \cdot 3 \cdot 2 \cdot 1]}{[4 \cdot 3 \cdot 2 \cdot 1]} = \frac{6!}{4!}$

2. Combinations, $C(n,k)$

Permutations should be contrasted with "combinations", for which "order doesn't matter". Consider the following tasks, where $X = \{a, b, c, d, e\}$ ($|X| = 5$):

(1) permutation: take 3 elts out of X and line them up:

$$P(5,3) = 5 \cdot 4 \cdot 3 = \frac{5!}{2!}$$

(2) Combination: take 3 elts out of X and form a subset of X:

→ # size-3 subsets of X

(# comb. of 3 things from X)

Note: order doesn't matter

$\{a,c,d\} = \{c,d,a\}$

$= ?$

We'll see a better way.

↑

Brute-force computation: $\{a,b,c\}$, $\{a,b,d\}$, $\{a,b,e\}$, $\{a,c,d\}$, $\{a,c,e\}$, $\{a,d,e\}$, $\{b,c,d\}$, $\{b,c,e\}$, $\{b,d,e\}$, $\{c,d,e\}$. $\boxed{10}$

The general question: How many combinations / subsets of k elts can we form out of a set (X) with n elts?

Def: We denote the above number by $C(n,k)$.

Let's try a "small case", with $n=4$, $k=2$, $X=\{a,b,c,d\}$.

We'll start with permutations and collapse the ones giving the same comb. into one.

The permutations

$P(4,2) = 4\cdot 3 = 12$

ab	ac	ad	bc	bd	cd	all the perm.
ba	ca	da	cb	db	dc	
↓	↓	↓	↓	↓	↓	"collapsing"
$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$	

Now we know that

$$C(n,k) = \frac{P(n,k)}{\#\text{ lists that each comb. is collapsed from}}$$

Note: # lists that get collapsed into each fixed subset (eg. $\{a, b\}$)

= # lists with k elts formed from the k elts in that fixed subset

$= P(k,k) = k!$

Conclusion: $C(n,k) \overset{(*)}{=} \dfrac{P(n,k)}{P(k,k)} = \dfrac{P(n,k)}{k!} \left(= \dfrac{\frac{n!}{(n-k)!}}{k!} = \dfrac{n!}{k!\,(n-k)!} \right)$

Another way to prove/explain (*):

it's equivalent to prove that $\underbrace{P(n,k)}_{T} \overset{(\star)}{=} \underbrace{C(n,k)}_{S_1} \cdot \underbrace{P(k,k)}_{S_2}$.

($\star$) follows from the mult. principle that to $\underbrace{\text{form an ordered list of } k \text{ elts from } n \text{ elts}}_{T}$, we can first $\underbrace{\text{select } k \text{ things from the elts}}_{S_1}$ and then order the k selected objects.

Another "larger" example: $C(5,3)$. 3 elts out of $\{a,b,c,d,e\}$.

permutations (known): abc, acb, bac, bca, cab, cba . . . . . . cde, ced, dce, dec, ecd, edc

every comb. comes from 6 perm.

"6 : 1 collapsing".

Combinations (want): $\{a,b,c\}$, $\{a,b,d\}$, . . . , $\{c,d,e\}$

$$C(5,3) = \frac{P(5,3)}{3!} = \frac{5\cdot 4\cdot 3}{3!} = 10.$$

Rules on notation and computation, now for $C(n,k)$:

- $C(n,k)$ is also often written as $\binom{n}{k}$ (read: n choose k)

- Recall that $P(n,k)$ can be computed in two ways

  (i) product of a descending seq. $n\cdot(n-1)\cdots(n-k+1)$ → fewer numbers, practical

  (ii) quotient of factorials $\frac{n!}{(n-k)!}$ → quotient of factorials elegant

(Consequently,) So can $C(n,k) = P(n,k)/k!$

(i)' $\binom{n}{k} = C(n,k) = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}$ → practical, quotient of two k-fold products

e.g. $\binom{5}{3} = \frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1} = 10.$

(ii)' $\binom{n}{k} = C(n,k) = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{k!\,(n-k)!}$ → elegant, quotient involving three factorials.

e.g. $\binom{5}{3} = \frac{5!}{3!\,2!}$

One fact that can be seen from (ii)' is that for any int $0 \le k \le n$,

$$\binom{n}{a} = \frac{n!}{a!(n-a)!}$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

e.g. $\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5!}{2!\,3!} = \binom{5}{2}$

A "better", non-algebraic explanation:

to choose which k elts to pick from n elts

is equiv. to choosing which $(n-k)$ elts to leave out.

So, often, to compute $\binom{n}{k}$ quickly, it's useful to compute $\binom{n}{a}$

where $a = \min(k, n-k)$ and use (i).

e.g. $\binom{7}{5} = \frac{7!}{5!\,2!}$

$= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \ldots$

$\binom{7}{5} = \binom{7}{2} = \frac{7 \cdot 6}{2 \cdot 1} = 21.$

e.g. $\binom{7}{4} = \binom{7}{3} = \frac{7 \cdot \cancel{6} \cdot 5}{\cancel{3 \cdot 2} \cdot 1} = 7 \cdot 5 = 35.$

$C(101, 99) = C(101, 2) = \binom{101}{2} = \frac{101 \cdot 100}{2 \cdot 1} = 101 \cdot 50 = 5050.$

3. Example problems (involving combinations)

1. How many size-4 subsets does $\{1, 2, 3, \cdots, 9\}$ have?

Answer: $\binom{9}{4} = \frac{9 \times \cancel{8} \times 7 \times 6}{\cancel{4} \times 3 \times \cancel{2} \times 1} = 9 \cdot 7 \cdot 2 = 126.$

2. How many 5-elt subsets of $\underbrace{\{1, 2, \cdots, 9\}}_{A}$ are there with exactly two even elts?

Soln: We can form such a subset by first picking two even number from the even number 2, 4, 6, 8 in A and then picking three odd numbers from the odd numbers 1, 3, 5, 7, 9 in A,

So the desired number $\binom{4}{2} \cdot \binom{5}{3} = \frac{4 \cdot 3}{2 \cdot 1} \cdot \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 6 \cdot 10 = 60.$

3. Take 5 cards out of a usual 52-card deck.

How many such hands are there with 2 clubs and 3 hearts?

(comb., order doesn't matter)

$$\binom{13}{2}\cdot\binom{13}{3} = \frac{13\cdot 12}{\cancel{2}\cdot 1}\cdot\frac{13\cdot\cancel{12}\cdot 11}{\cancel{3\times 2}\times 1} = 13^2\cdot 12\cdot 11.$$

4. How many 7-digit binary strings (eg 0010011) have an odd number of 1's?

"How to make combinations appear?" Try a smaller example:

4-digit binary string, three 1's.

1110, 1101, 1011, 0111.

$\Updownarrow$ positions of the 1s.

$\{1,2,3\}$ $\{1,2,4\}$ $\{1,3,4\}$ $\{2,3,4\}$ $\rightarrow \binom{4}{3}$.

Soln: For any fixed int $0 \le k \le 7$, the number of 7-digit strings with k 1s should be $\binom{7}{k}$, corresponding to the positions of the k 1s in the string.

It follows that the desired number is

$$\binom{7}{1} + \binom{7}{3} + \binom{7}{5} + \binom{7}{7}$$

(since we want an odd number of 1s)

$$= 7 + \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} + \binom{7}{2} + 1$$

$$= 7 + 35 + 21 + 1$$

$$= 64. \quad \square$$

Note: The number 64 is precisely half of all possible 7-digit binary strings ($2^7 = 128$); equivalently, there are as 7-digit strings with an even number of ones (indeed, there are $\binom{7}{0}+\binom{7}{2}+\binom{7}{4}+\binom{7}{6} = 64$).

"Why really?" Is there an explanation without manipulating formulas?

$$\binom{7}{1}+\binom{7}{3}+\binom{7}{5}+\binom{7}{7} = \binom{7}{0}+\binom{7}{2}+\binom{7}{4}+\binom{7}{6}$$

tomorrow: "combinatorial identities".