# Math 2001. Lecture 5.

06.07.2022.

**Last time:**

- Logical equivalences
- Open sentences and quantifiers → finishes Ch. 2.

**Today:**

Ch 3: counting

- the multiplication, addition, and subtraction principles
- practice counting problems

# 1. The multiplication principle

The principle: Suppose a task $T$ can be completed in $n$ independent steps $S_1, S_2, \ldots, S_n$, and suppose that each step $S_i$ can be done in $a_i$ ways. Then $T$ can be performed in $a_1 a_2 \cdots a_n$ ways.

(what's done in one step does not affect choices in other steps)

E.g. (3.2) In ordering a latte, you have a choice of $\underbrace{\text{whole, skim or soy milk}}_{S_1}$; $\underbrace{\text{small, medium or large}}_{S_2}$; and either $\underbrace{\text{one or two shots of espresso}}_{S_3}$. How many choices do you have for ordering one latte?

Answer: There are $\underset{a_1}{3} \cdot \underset{a_2}{3} \cdot \underset{a_3}{2} = 18$.

Rmks: · We have in fact used the principle before, for Cartesian products of sets, # subsets of a set, and the menu and dice examples from Lecture 1.

· Sets offer a nice framework for counting problems and principles. Indeed, the multiplication principle is equiv. to the fact that

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|$$

Practical point: Sometimes it's useful to formulate counting problems in set language, and it's useful to count things by formulating them as results of tasks that can be performed in independent steps ("dynamic processes", so that we can use the mult. principle, e.g. counting power sets).

## 2. The addition and subtraction principles (in terms of sets)

no two of the subsets intersect

The addition principle: If a set $X$ is the pairwise disjoint union of a number of subsets $X_1, X_2, \ldots, X_n$, then

$$|X| = |X_1| + |X_2| + \cdots + |X_n| = \sum_{i=1}^{n} |X_i|$$

The subtraction principle: For a set $X$ in a finite universe $\mathcal{U}$, we have

$$|X| = |\mathcal{U}| - |\overline{X}| .$$

"Sometimes it's easier to count the opposite kind of things / configurations".

# 3. Counting problems.

**Note:** Tuples / lists are always ordered in this course.

1. Consider lists of length 4 made with symbols A, B, C, D, E, F.

(a) How many such lists are possible if repetition of the letters is allowed?

(e.g. ABCB, EDFB, EDFB $\neq$ DEFB)

T: _ _ _ _ specify the four letters in the four positions

$S_i$ ($i = 1, 2, 3, 4$): specifying the ith letter

Soln: Every entry in the list can be specified independent of the other entries in 6 ways, so there are $6 \cdot 6 \cdot 6 \cdot 6 = 6^4 = 1296$ possible such lists.

wb) What if repetition is not allowed? _ _ _ _

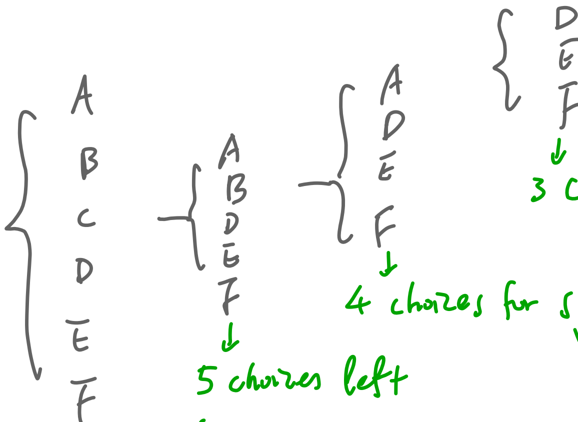
T: same, but the letters should be specified without rep.

$S_i$ ($i = 1, 2, 3, 4$): fill in the ith spot with one of the remaining letters.

<u>Soln</u>: There are 6 choices for the first spot, and <u>then</u> 5 choices for the second spot (no matter what went to the first spot) and then 4 choices for the third and 3 choices for the last spot. So there are

$6 \times 5 \times 4 \times 3 = 360$ such lists.

A picture for (b):




3 choices for spot 4.

4 choices for spot 3

5 choices left for spot 2

6 choices

1.(c) How many such lists are possible if rep. is not allowed and the list has an E? (e.g. E A C F , C F E A , ~~BDAF~~)

$\underline{S_1}$: specify which one of the four spots gets an E

$\rightarrow a_1 = 4.$

$\underline{S_2}$: fill in the remaining 3 spots with (3 of) the five letters left.

similar to (b)

_ _ _ $\rightarrow a_2 = 5 \cdot 4 \cdot 3.$

$\underline{\text{Soln}}$: There are 4 choices for where to put E and then $5 \cdot 4 \cdot 3$ choices for filling the rest of the list, so there are $4 \cdot (5 \cdot 4 \cdot 3) = 240$ such lists.

1(d). How many such lists are there if rep. is allowed and the list must have an E? (e.g. AEBF, EECE, ...)
(at least one E)

Suggestion: $4 \cdot 6^3$

$4$: specify where an E goes

$6^3$: specify the rest

issue: this word is counted multiple times.
specified

the number $4 \cdot 6^3$ would be an overcount and too large.

Soln: We can use the subtraction principle:

the desired number (rep allowed, E included)

= total number of words (with rep. allowed)

− # words (rep allowed, E excluded)

$= 6^4 - 5^4$.

Let $S$ be the set of 4-letter words, rep allowed, using letters from A, B, C, D, E, F.
Then the desired number is

$|S| - |\{\underline{w} \in S : \underline{w} \text{ has no } E\}|$

$= 6^4 - 5^4$.

2. Make a non-repetitive list of length 5 from the symbols A, B, C, D, E, F, G.

(a) How many such lists are there if the first entry must be B, C, or D and the last entry must be vowel?

fill in the remaining 3 spots with the 5 letters left

_ _ _ _ _

B, C, D      A, E.

Soln: There are $3 \cdot 2 \cdot \underline{5 \cdot 4 \cdot 3} = 360$.

choose the 1st letter

last letter

(b) What if the last letter should be a vowel and the first letter is A, C, or D?

_ _ _ _ _

A, C, D      A, E.

Soln: Depending on whether the first letter is A, there are $5 \cdot 4 \cdot 3 + 2 \cdot 2 \cdot 5 \cdot 4 \cdot 3 = 5 \cdot 5 \cdot 4 \cdot 3$ such lists.

(first letter is A

A _ _ _ E)

(first letter is not A

C, D _ _ _ A, E)

disjoint

What if we discussed the cases based on what the last letter is?

Case 1: The last letter is A.

$\underset{C, D}{\_\_}\ \_\_\ \_\_\ \_\_\ A$

$\to 2 \cdot 5 \cdot 4 \cdot 3.$

Case 2. $\_\ \_\ \_$ is not A, and hence is E.

"disjoint" cases

$\underset{A, C, D}{\_\_}\ \_\_\ \_\_\ \_\_\ E$

$\to 3 \cdot 5 \cdot 4 \cdot 3$

Total: $2 \cdot 5 \cdot 4 \cdot 3 + 3 \cdot 5 \cdot 4 \cdot 3 = 5 \cdot 5 \cdot 4 \cdot 3.$ ✓

## General strategies:

- use the mult. principle to simplify things when possible.
- use the addition principle for cases, making sure not to overcount.
- recognize when the subtraction principle may be useful.
- Overall, do not miss objects and do not overcount.

E.g. (Ex 3.2.5.) How many integers between 1 and 9999 have at least one repeated digit? (e.g. 747.)

$$9999 - (\underbrace{9 + 9\cdot 9 + 9\cdot 9\cdot 8 + 9\cdot 9\cdot 8\cdot 7}_{\text{no rep, based on \# digits}})$$

## More problems :

3. Five cards are taken out of a standard 52-card deck and lined up in a row.

order matters

(a) How many such line-ups are there with at least one red card?

Answer : $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 - \underbrace{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}_{\text{no red, i.e., all black}}$.

(b) . . . . . . . if the cards are all blacks or all hearts?

$$\underbrace{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}_{\text{all blacks}} + \underbrace{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}_{\text{all hearts}}$$

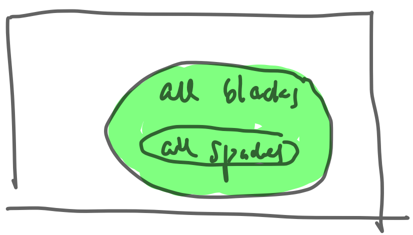
disjoint choices

(c) What if the cards are to be of the same color?

$$\underbrace{26\cdot 25\cdot 24\cdot 23\cdot 22}_{\text{all blacks}} + \underbrace{26\cdot 25\cdot 24\cdot 23\cdot 22}_{\text{all reds}}$$

(d) - - - - - - - - , all blacks or all spades?

$$26\cdot 25\cdot 24\cdot 23\cdot 22.$$




if $B\subseteq A$, then $A\cup B=A$ and hence

$$|A\cup B| = |A|.$$

Contrast with 3(b) and the addition principle, where $A\cap B=\phi$, $A\cup B\neq A$, and $|A\cup B| = |A|+|B|$.