Math 2001. Lecture 4.

Midterm 1: this Friday, June 10.

06.06.2002.

Last time:

- indexed sets
- statements, truth values, and truth tables (including compound and conditional statements)
- strategy for proving various kinds of statements.

  e.g. To prove $p \Rightarrow q$, assume p and deduce q.

Today:

- logical equivalence
- open statements and quantifiers

1. Logical equivalences

We say two statements are (logically) equivalent if they always have the same truth value (no matter what the truth values of their component statements are).

Non-example: $(P \Rightarrow Q)$ vs $(Q \Rightarrow P)$ (converses of each other)

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

- vacuously true: $X \Rightarrow Y$ is true if X is false.

$(P \Rightarrow Q)$ is not equiv. to $(Q \Rightarrow P)$ $\Leftarrow$

Example. " $P \Leftrightarrow Q$ " is logically equiv. to " $(P \wedge Q) \vee ((\sim P) \wedge (\sim Q))$ ":

$\underbrace{P \Leftrightarrow Q}_{X}$ $\qquad$ $\underbrace{(P \wedge Q) \vee ((\sim P) \wedge (\sim Q))}_{Y}$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$	$P \wedge Q$	Y
T	T	T	T	T	F	F	F	T	T
T	F	F	T	F	F	T	F	F	F
F	T	T	F	F	T	F	F	F	F
F	F	T	T	T	T	T	T	F	T

same in all rows,
so X and Y are equivalent.

Example. "$P \Rightarrow Q$" is equivalent to "$\sim Q \Rightarrow \sim P$"

X — Y, the "contrapositive" of X.

Later: we may prove a conditional statement by proving its contrapositive by the above equivalence.

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

✓

Exercise: "$P \vee Q$" is equivalent to "$(\sim P \Rightarrow Q) \wedge (\sim Q \Rightarrow P)$".

Exercise: (DeMorgan's Laws) Let P, Q be statements. Then

(i) $\sim(P \wedge Q) = (\sim P) \vee (\sim Q)$

(ii) $\sim(P \vee Q) = (\sim P) \wedge (\sim Q)$.

("=" : equivalent)

Compare (i) and (ii) with DeMorgan's Laws for sets

(i)' $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(ii)' $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Rmk: We can prove the set version (i)', (ii)' of DeMorgan's Laws from the logic version (i), (ii) easier:

Imagine a universe $\mathcal{U}$ and suppose $A \subseteq \mathcal{U}$, $B \subseteq \mathcal{U}$.

For all $x \in \mathcal{U}$, let P be the statement "$x \in A$" and Q the statement "$x \in B$". Then $P \wedge Q$ = "$x \in A \cap B$" by the def of set intersections and $P \vee Q$ = "$x \in A \cup B$" by the def of unions.

Now, by (i), for all $x \in \mathcal{U}$ we have $\sim(P \wedge Q) = (\sim P) \vee (\sim Q)$,

so "$x \notin A \wedge B$" = "$(x \notin A)$ or $(x \notin B)$",

i.e., "$x \in \overline{A \cap B}$" = "$x \in \bar{A} \vee x \in \bar{B}$"

i.e., "$x \in \overline{A \cap B}$" = "$x \in \bar{A} \cup \bar{B}$".

It follows that $\overline{A \cap B} = \bar{A} \cup \bar{B}$, i.e. (i)' holds.

Ex. Prove that (ii) $\Rightarrow$ (ii)'.

2. Open sentences and quantifiers

E.g. "$\underbrace{x \text{ is an odd integer}}_{P(x)}$" is not a statement: we can only judge its truth value after x is made more specific.

We call such sentences open statements.

We need quantifiers to make it a statement.

There are two main quantifiers in math:

- the existential quantifier "for some", written "$\exists$".

  e.g. For some $x \in \mathbb{R}$, x is an odd integer. $\to$ True statement.

  $\exists x \in \mathbb{R}$ s.t. $P(x)$. $\leftarrow$ "$\exists\, x \in \mathbb{R}$ such that x is an odd integer."

- the universal quantifier "for all", (s.t.) written "$\forall$".

  $\forall x \in \mathbb{R}$, $P(x)$. $\leftarrow$ e.g. $\forall\, x \in \mathbb{R}$ / For all $x \in \mathbb{R}$, x is an odd int. $\to$ false statement.

Rmk: • Note that given a universe U and an open sentence $p(x)$, the statement " $\forall x \in U,\ p(x)$ " is equivalent to " $x \in U \Rightarrow p(x)$ ".

• Make sure you use quantifiers when necessary!

Avoid "any" and use "every/each" or "some" to avoid ambiguity when ambiguity is possible.

Examples: Write the following statements in English. Then determine if they are true or false.

(1). $\exists x \in \mathbb{R},\ x^2 > 0$.

T: e.g. $x = 1$ is such an example.

For some $x \in \mathbb{R}$, we have x^2 is positive

There exists some real number whose square is positive.

(2) $\forall x \in \mathbb{R},\ x^2 > 0$.

For every / all real number x, we have $x^2 > 0$.

Every real number squares to a positive number.

False: $x = 0$ gives a counter example.

(3) $\exists a \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R},\ ax = x$.

There is a real number a s.t. ax equals x for every real number x.

True: 1 is such a number.

(4) $\forall n \in \mathbb{Z},\ \exists m \in \mathbb{Z}$ s.t. $m = n+5$.

True, (ultimately) since the sum of two int. is an int.

Ex: put these in English.

(5) $\exists m \in \mathbb{Z}$ s.t. $\forall n \in \mathbb{Z},\ m = n+5$. $\rightarrow$ False, since, for example, $1+5 \neq 2+5$.

Note:

- We can prove existential claims $\exists x (\in U)$ s.t. $P(x)$ by producing an example but can't do the same for universal claims $\forall x \in U, P(x)$.

  (e.g. for $x=1$, we do have $x^2>0$, but this doesn't make $\underline{\underline{\forall}} x \in \mathbb{R}, x^2>0$.)

- On the other hand, we can disprove a universal claim $\forall x \in U, P(x)$ by finding a counter-example, i.e. finding one $x \in U$ s.t. $P(x)$ fails.

  (e.g. we found $x=0$ in (2)).

- To prove a universal claim $\forall x \in U, P(x)$, we often prove it as the conditional statement $x \in U \Rightarrow P(x)$, i.e. we assume x is a general element in U and establish $P(x)$.