Math 2001. Lecture 3.

06. 03. 2022.

Last time:

- set operations: unions, intersections, difference, complements

  not commutative

  $A-B \neq B-A$.

- Venn diagrams: visualizing set interactions and operations

- Proofs of set equalities, e.g. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

DeMorgan's Laws

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

Today:

- proof of DeMorgan's Laws, Indexed sets
- Ch 2: Statements (logic) $\left\{ \right.$ compound statements, truth tables, how to prove various types of statements

o. DeMorgan's Law

Q: $\overline{A \cup B \cup C} = \bar{A} \cap \bar{B} \cap \bar{C}$?
$\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$?

We will prove $\overline{A \cup B} = \bar{A} \cap \bar{B}$ (to not be in either A or B = to be in neither A nor B):

LHS: $\overline{A \cup B}$, RHS: $\bar{A} \cap \bar{B}$

the other half ($\overline{A \cap B} = \bar{A} \cup \bar{B}$) has a similar proof.

Pf: (1) LHS $\subseteq$ RHS. Let $x \in \overline{A \cup B}$. Then x is not in either A or B.

It follows that $x \notin A$ and $x \notin B$. So $x \in \bar{A}$ and $x \in \bar{B}$.

So $x \in \bar{A} \cap \bar{B}$. Therefore $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

(2) RHS $\subseteq$ LHS. Let $x \in \bar{A} \cap \bar{B}$. Then $x \in \bar{A}$ and $x \in \bar{B}$, so $x \notin A$ and $x \notin B$.

It follows that $x \notin A \cup B$.

So $x \in \overline{A \cup B}$. Therefore $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

By (1) and (2), we conclude that $\overline{A \cup B} = \bar{A} \cap \bar{B}$. $\square$

1. Indexed sets

Motivation: Sometimes we deal with a large collection of sets interacting in some way all at once. We need efficient notation.

Eg. Given a collection of sets $A_1, A_2, \cdots, A_9, A_{10}, \cdots$

We may write $\bigcup_{i=1}^{6} A_i$ as a shorthand for the union

index set $= \{1, 2, \cdots, 6\}$

"index"

for $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$;

we may write $\bigcap_{i=1}^{10} A_i$ for the intersection

index set $= \{1, 2, \cdots, 10\}$

$A_1 \cap A_2 \cap A_3 \cap \cdots \cdots \cap A_{10}$.

Note: The indices form a set themselves.

Def: For an index set I, we define

$$\bigcup_{i\in I} A_i = \{ x \mid x \in A_i \text{ for } \underline{\text{some}} \ i \in I \}$$

" quantifiers "

$$\bigcap_{i\in I} A_i = \{ x \mid x \in A_i \text{ for } \underline{\text{all}} \ i \in I \}$$

Examples:

- If $A_1 = \{-1, 0, 1\}$, $A_2 = \{-2, 0, 2\}$, $A_3 = \{-3, 0, 3\}$, $\cdots$, $A_n = \{-n, 0, n\}, \cdots$

Let $I = \{1, 2, 3, \cdots\} = \mathbb{Z}_{\geq 1}$.

Then $\bigcup_{i\in I} A_i = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} = \mathbb{Z}$

and $\bigcap_{i\in I} A_i = \{0\}$

- If $A_n = \{0, 1, 2, 3, \dots, n\}$ for each $n \in \mathbb{Z}_{\geq 0}$, then

$$\bigcup_{i=0}^{\infty} A_i = \mathbb{Z}_{\geq 0}$$

$$\bigcup_{i \in I} A_i \text{ for } I = \{0, 1, 2, \dots\}$$

$$\bigcap_{i=1}^{\infty} A_i = \{0\}$$

$A_0 : \{0\}$

$A_1 : \{0, 1\}$

$A_2 : \{0, 1, 2\}$

$A_3 : \{0, 1, 2, 3\}$.

Note: For any collection of sets A_α with index set I, we have

$$\bigcap_{\alpha \in I} A_\alpha \subseteq A_\beta \subseteq \bigcup_{\alpha \in I} A_\alpha \quad \text{for any } \beta \in I$$

by the definitions of intersections and unions of sets.

2. Statements (§ 2.1)

A statement is a sentence or mathematical assertion that is either definitely true or definitely false.

(as opposed to quantities, expressions, opinions, questions, ...)

Example and non-examples

- $2 \in \mathbb{Z}$ (The number 2 is an integer.) $\longrightarrow$ a statement; a true one
- $\pi \in \mathbb{Z}$ $\longrightarrow$ a stat e ent, but a false statement
- $\mathbb{Z} \subseteq \mathbb{R}$ $\longrightarrow$ a true statement
- 42 $\longrightarrow$ not a statement, no assertion is made

- "What is the solution of $2x = 84$?" → a question, not a statement

- Add 3 and 5. → An instruction, not a statement.

- "If n is an even integer, then $(n+1)$ is an odd integer."

  → This is a statement. It's an "if-then" statement and also called a conditional statement.

  And it's true.

- "If n is an odd integer, then $2n$ is an odd integer."

  → a false statement, counterexample: $n=1$ is odd, but $2 \cdot 1 = 2$ is even.

- "$2 \in \mathbb{Z}$ and $\pi > 3$". → a statement, a compound statement. and it's true.

3. New statements from old: and, or, not

We often combine statements into more complex ones.

We are interested in how the validity of the simple statement affect the validity of the new statement.

Notation: "T": true, "F": false

- We'll often denote statements by letters such as P, Q, R, ...
- We write "$\wedge$" for "and", "$\vee$" for "or", and "$\sim$" for "not".

e.g. "$\sim(2 \in \mathbb{Z})$" $= 2 \notin \mathbb{Z}$. → false

$(2 \notin \mathbb{Z}) \wedge (3 \in \mathbb{Z})$ → false (since $2 \in \mathbb{Z}$)

Truth tables: We often use truth tables to indicate how the truth value / validity of a complex statement depends on those of its components

"and" / $\wedge$:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

"or" / $\vee$:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

"not" / $\sim$

P	$\sim P$
T	F
F	T

More interestingly, we can deduce the truth value of more complex statements once we know the truth values of its component statements.

E.g. Suppose we have

P	Q	R	S	O
T	F	T	F	F

.

Then $$\underline{P} \vee \underline{Q} \vee \underbrace{\Big(\underset{T}{R} \wedge \underbrace{\big(\underset{F}{S} \vee \underset{T}{\underline{(\sim O)}}\big)}_{T}\Big)}_{T}$$ is $\underline{\ T\ }$.

4. New statements from old: conditional statement

By a conditional statement we mean a statement that can be made in the form "if P, then Q" where P, Q are themselves statements.

Running example: if $\underbrace{x \text{ is an integer divisible by } 6}_{P}$, then $\underbrace{x \text{ is an even int}}_{Q}$.

Note: A conditional statement can often be stated in various ways:

(P44)

notation for "if P then Q": $P \Rightarrow Q$

- if P, then Q;
- Q if P.
- Q whenever P (whenever P, Q)
- Q, provided that P holds.
- P implies Q.
- P guarantees Q
- P is sufficient / a sufficient condition for Q
- ✡ Q is necessary / a necessary condition for P.
- ✡ P only if Q.

To prove an "if P, then Q" statement, show that Q holds whenever P holds. (In our example, we need to show that x is an even integer whenever x is an int. divisible by 6.)

Note: We just need to deduce Q in the cases where P happens.

($x=5$, is not even → this doesn't affect our proof in the example because 5 is not divisible by 6.)

The proof template should be:

"Suppose P holds. Then [observations and deductions] ...

... ... So Q holds

Therefore P implies Q."

Pf of our example statement:

Suppose x is an integer divisible by 6.

Then $x = 6 \cdot n$ for some integer n.

Thus, we have $x = (2 \cdot 3) \cdot n = 2 \cdot (3n)$ where $3n \in \mathbb{Z}$ since $n \in \mathbb{Z}$.

It follows that x is an even int.

Therefore our statement holds. $\square$

Remarks: (1). We define the converse of a conditional statement "if P then Q" ($P \Rightarrow Q$) to be the statement "if Q then P". ($Q \Rightarrow P$)

Note that "$P \Rightarrow Q$" and "$Q \Rightarrow P$" generally have different truth values.

Eg. x is an int divisible by 6 $\Rightarrow$ x is an even int : True

x is an even int $\Rightarrow$ x is an int divisible by 6. : False

e.g. $x=4$.

(2) Summary: to prove $P \wedge Q$, prove both P and Q.

... $P \vee Q$, prove one of P and Q.

... $\sim P$, prove P is false.

... $P \Rightarrow Q$, assume P and deduce Q.

... $P \Leftrightarrow Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$ separately.