Math 2001. Lecture 22. 06. 30. 2022.

Last time:

- partitions from equivalence relations
- functions: domain, codomain, range / image
  inj., surj., bij.,
  composition

Today:

- problems on function
- review for the final exam

1. Problems on functions

1. Prove that (1) if $f: A \to B$ and $g: B \to C$ are both inj, then the composition $g \circ f$ $\to (A \to C)$ is also inj.

(Thm 12.2)

E.X.

(2) The above also holds if we replace both "inj." by "surj".

Pf of (1): (Recall that a function $\varphi: X \to Y$ is inj if

$$\underline{\varphi(x_1) = \varphi(x_2) \Longrightarrow x_1 = x_2} \quad \forall x_1, x_2 \in X.)$$

Let $a_1, a_2 \in A$. Suppose f, g are inj, and suppose that

$$(g \circ f)(a_1) = (g \circ f)(a_2).$$

By def. of compositions. this implies that $g(\underline{f(a_1)}) = g(\underline{f(a_2)})$.

Since g is inj, it follows that $f(a_1) = f(a_2)$. Since f is inj, it further follows that $a_1 = a_2$. Therefore $f \circ g$ is inj.

2. Are the following functions inj ? ... surj ?

(1) $f: \mathbb{Z} \to \mathbb{Z}$, $f(n) = 2n+1 \quad \forall n \in \mathbb{Z}$.

inj ? $\quad f(n_1) = f(n_2) \overset{?}{\Longrightarrow} n_1 = n_2$

$2n_1 + 1 = 2n_2 + 1$

Yes, by basic algebra

surj ? $\quad im(f) = \{f(n) : n \in \mathbb{Z}\} = \{2n+1 : n \in \mathbb{Z}\} \overset{?}{=} \mathbb{Z}$.

No, such values are all odd.

(2) $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $f(m,n) = 8m + 5n \quad \forall m,n \in \mathbb{Z}$.

inj ? $\quad 8m_1 + 5n_1 = 8m_2 + 5n_2 \overset{?}{\Longrightarrow} (m_1, n_1) = (m_2, n_2)$

$\forall m_1, n_1, m_2, n_2 \in \mathbb{Z}$

No, for example we have $(5,0) \neq (0,8)$ but

$f(5,0) = f(0,8) = 40$.

Surj?

$f(m,n) = 8m + 5n.$

$\operatorname{Im} f = \{ 8m+5n \mid m,n \in \mathbb{Z} \} \stackrel{?}{=} \mathbb{Z}.$

Yes: since $\gcd(8,5) = 1$, we can write 1 as $8m+5n$ for some $m,n \in \mathbb{Z}$. Indeed, we have $1 = 8\cdot 2 + 5\cdot(-3)$.

Thus, given any $k \in \mathbb{Z}$, we have

$$k = 1\cdot k = \big(8\cdot 2 + 5\cdot(-3)\big)\cdot k$$

$$= \underbrace{8\cdot(2k) + 5\cdot(-3k)}_{\in \operatorname{Im}(f)}$$

where $2k, -3k \in \mathbb{Z}$. It follows that $\mathbb{Z} = \operatorname{Im}(f)$, so f is surj.

3. Consider functions $f: \{A, B, C, D, E, F, G\} \to \{1, 2\}$.

(1) How many such functions are there?

(2) - - - - - - inj?

(3) - - - - - - surj? $\to$ Use the subtraction principle:

surj. functions

= # all functions − # non-surj functions.

all outputs = 1 or all outputs = 2.

$= \boxed{2^7 - 2}$

Soln: (1) To specify such a function is to assign an output equal to either 1 or 2 (2 choices) to each of the seven elts in the domain. The assignments can be made independently, so there are 2^7 possible functions in total.

(2) None, since $|\{A, B, C, D, E, F, G\}| > |\{1, 2\}|$.

(4) - - - - - - bij?

(4). None, since none are inj by (2).

4. Consider functions $f: \{A, B, C, D, E\} \to \{1, 2, 3, 4, 5\}$.

(1) How many such functions are there?

(2) - - - - - - inj?

(3) - - - - - - surj?

(4) - - - - - - bij?

permutations of 1, 2, 3, 4, 5.

e.g. 31245

$$f: \begin{array}{l} A \to 3 \\ B \to 1 \\ C \to 2 \\ D \to 4 \\ E \to 5. \end{array}$$

Answer: (1). 5^5 (2), (3), (4): $5!$

Ex.: Write down the solns (with complete reasonings).

2. Review

1. Key notions and notations.

- sets, set builder notation $\{ \text{objects/expressions} : \text{properties} \}$.

  e.g. $\{ n \in \mathbb{Z} : 2 \mid n \}$,

  or $\{ n \mid n \in \mathbb{Z}, 2 \mid n \}$.

- Constructions on sets (new from old):

  Cartesian products, power set, unions, intersections, difference, complement

- Venn diagrams (T/F problem about set equalities)

- statement, truth tables, logical equivalence.
  conditional statements, converses, contrapositives (a conditional statement is always equiv. to its contrapositive, but not to its converse)
  negations, quatifiers (use them when necessary!)

- permutations, combinations, and their variations involving multisets.
  (building correct models for bars-and-stars method)

  factorials, binomial coefficients, Pascal's triangle, binomial thm.

  inclusion-exclusion principle.

- pigeonhole principle, division principle ("worst-case scenarios")

- (for proofs) congruence of integers (mod a positive number n),
  integer division, (characterization) of gcd's.

- induction and strong induction.

- relations and partitions, functions

  → source for basic proof problems using definitions

2. Key techniques

Counting:
- addition, subtraction, and multiplication principles.
  "dynamic counting" / "algorithmic thinking"
- inclusion-exclusion
- bars-and-stars method.

proofs: know when and how to

- prove (bi)conditional statements (including set containments and equalities)
- prove by cases / contradiction / contrapositive
- use (strong) mathematical induction (use recursions!).

$\square$