Math 2001. Lecture 21.

06. 29. 2022.

Last time:

- relations R on sets A and their (potential) properties
  - reflexive: $xRx \ \forall x \in A$
  - symmetric: $xRy \Rightarrow yRx \quad \forall x, y \in A$
  - transitive: $xRy \wedge yRz \Rightarrow xRz \quad \forall x, y, z \in A$
- equivalence relations: rels that are refl., symm., and transitive.

  $\downarrow$

  equivalence classes $\{ [a] := \{ x \in A \mid xRa \} \mid a \in A \}$
- Verifications of equivalence relations (e.g. congruence, fractions / rational numbers)

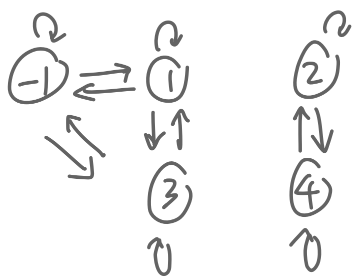
Today:

- partitions from equivalences
- functions (Ch. 12).

1. Partitions from equivalence rels.

Def: A <u>partition</u> of a set A is a set / collection of nonempty subsets of A s.t. (1) no two of them intersect nontrivially, i.e., the sets are pairwise disjoint ; and (2) their union equals A.

Eg. Yesterday we saw that the equivalence classes of the rel.



on $A := \{-1, 1, 3, 2, 4\}$ are

$$[-1] = [1] = [3] = \{-1, 1, 3\}$$

$$[2] = [4] = \{2, 4\}.$$

These two classes $[-1] = [1] = [3]$ and $[2] = [4]$ $\underbrace{\text{form a partition of}}_{\text{partition (verb)}}$ A.

Two theorems:

Thm 1. Let R be an equivalence rel. on a set A. Then $\forall a, b \in A$, we have $[a] = [b] \iff aRb$.

Thm 2. Let R be an equivalence rel. on a set A. Then the (distinct) equivalence classes of A $(\{[a] : a \in A\})$ partition A.

Pf of Thm 1: Suppose / Let $a, b \in A$.

(1) $(\Rightarrow)$ Suppose $[a] = [b]$. By def, we have $[b] = \{x \in A : xRb\}$, so since $a \in [a] = [b]$, where $a \in [a]$ since R is reflexive, we have aRb.

(2) $(\Leftarrow)$ Suppose aRb. We need to show that $[a] = [b]$. We do so by showing that $[a] \subseteq [b]$ and $[b] \subseteq [a]$.

(i) We prove $[a] \subseteq [b]$ first. Take $x \in [a]$. Then xRa.

Since xRa and aRb by assumption and R is transitive, we have xRb.

Therefore $x \in [b]$, so $[a] \subseteq [b]$.

(ii) We also need $[b] \subseteq [a]$. This follows from the fact that bRa, which in turn holds since aRb and R is symm.

By (i), (ii), we see that $aRb \Rightarrow [a] = [b]$.

By (1), (2), we have $aRb \Leftrightarrow [a] = [b]$. $\square$

Pf of Thm 2: Let $\mathcal{E} = \{ [a] : a \in A \}$. We need to show that $\mathcal{E}$ forms a partition of A. To do so, we note:

(1). Suppose that $a, b \in A$ are s.t. $\underbrace{[a] \neq [b]}_{(*)}$. We claim that $[a] \cap [b] = \phi$.

(Pf of the claim, by contradiction) Otherwise $\exists\, c \in A$ st. $c \in [a]$ and $c \in [b]$.

Then cRa and cRb. Since cRa and R is symm, we have aRc.

Since aRc, cRb and R is transitive, we have aRb.

But then we must have $[a] = [b]$ by Thm 1, contradicting $(*)$.

So $[a] \cap [b] = \phi$, so the sets in $\mathcal{E}$ are pairwise disjoint.

(2) To show that A is the union of the sets $[a]$ $(a \in A)$ in $\mathcal{E}$, it suffices to show that every elt $a \in A$ is in one equivalence class of R. This is true since $\underbrace{a \in [a]}_{\text{since } A \text{ is reflexive}}$, the class of a itself.

So A is indeed the union of the sets in $\mathcal{E}$.

By (1) and (2), we conclude that $\mathcal{E}$ forms a partition of A, as desired. $\square$

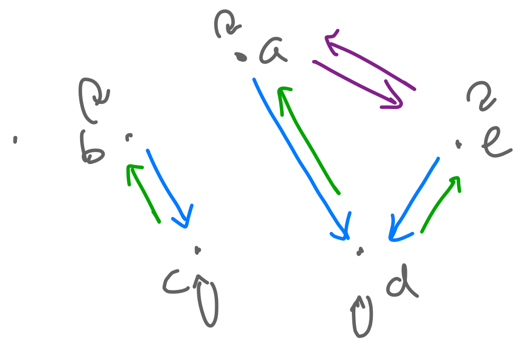
Example: Let $A = \{a, b, c, d, e\}$. Suppose R is an equiv. rel. on A.

Suppose R has two equiv. classes. Also, suppose, aRd, bRc, and eRd. Write out R as a set.

$aRb \leftrightarrow (a,b) \in A \times A$.

Exercise: write out the argument in words.

Analysis via the rel. graph:



loops: reflexivity.

$\rightarrow$: given info.

$\rightarrow$: symmetry & $\rightarrow$

$\rightarrow$: transitivity & $\rightarrow$ $\rightarrow$

the two connected components drawn in the graph must correspond precisely to the two classes, $\{b, c\}$, $\{a, d, e\}$. $\Leftarrow$

# of classes = 2

2. Functions

Def. (Def 12.1) A function f from a set A to a set B is a rule that assigns one (unique) elt in B to each elt in A.

If an elt $a \in A$ is assigned the output $b \in B$, we write $f(a) = b$.

Def: For a function $f : A \to B$, we call A the domain of f and B the codomain of f. The range or image of f is the set $\operatorname{im}(f) = \{ f(a) : a \in A \} \subseteq B$.

Note: In general $\operatorname{im}(f)$ may not equal the entire codomain, i.e., we may have $\operatorname{im}(f) \subsetneq B$. eg. For $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, we have

$$\operatorname{im}(f) = \{ x^2 : x \in \mathbb{R} \} = \mathbb{R}_{\geq 0} \subsetneq \text{the codomain } \mathbb{R}.$$

Def (Def 12.3) Two functions $f: A \to B$ and $g: C \to D$ are <u>equal</u> if $A = C$ and $f(x) = g(x)$ $\forall x \in A = C$.

Eg. $\left(f: \mathbb{R} \to \mathbb{R}, f(x) = x^2 \; \forall x \in \mathbb{R} \right) = \left(g: \mathbb{R} \to \mathbb{R}_{\geq 0}, g(x) = x^2 \; \forall x \in \mathbb{R} \right)$

Def (Def 12.5) Suppose $f: A \to B$ and $g: B \to C$ are functions. Then <u>the composition of f and g</u> is the function $g \circ f : A \to C$ given by

$$(g \circ f)(a) = g(\underbrace{f(a)}_{B}) \quad \forall a \in A.$$

Eg. If $f, g: \mathbb{R} \to \mathbb{R}$ are given by $f(x) = x+1$ and $g(x) = x^2$ $\forall x \in \mathbb{R}$. Then

$$(g \circ f)(x) = g(x+1) = (x+1)^2 = x^2 + 2x + 1 \quad \forall x \in \mathbb{R}$$

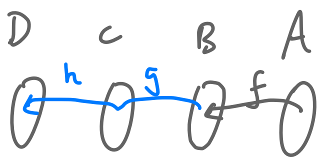
and $(f \circ g)(x) = f(x^2) = x^2 + 1$.

Note: $f \circ g \neq g \circ f$ in general, even if both $f \circ g$, $g \circ f$ are defined.

Thm. (Thm 12.5) Composition of functions is associative, that is, if $f: A \to B$, $g: B \to C$, $h: C \to D$ are functions, then

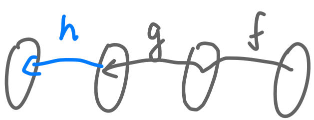
$$(h\circ g)\circ f = h\circ(g\circ f).$$

Pf: Both $(h\circ g)\circ f$ and $h\circ(g\circ f)$ have domain A, so it suffices to show that $[(h\circ g)\circ f](a) \overset{(*)}{=} [h\circ(g\circ f)](a)$ $\forall a\in A$. Let $a\in A$. Then



$$[(h\circ g)\circ f](a) = h\circ g\big(f(a)\big) = h\big(g(f(a))\big)$$

$$\|$$



$$[h\circ(g\circ f)](a) = h\big(g\circ f(a)\big) = h(g(f(a)))$$

So $(*)$ holds for all $a\in A$, therefore $(h\circ g)\circ f = h\circ(g\circ f)$. $\square$

Def. (Def 12.4) A function $f: A \to B$ is

(1) <u>injective</u> (or <u>one-to-one</u>) if $\forall a, a' \in A$, we have $f(a) = f(a') \Rightarrow a = a'$.

contrapositives

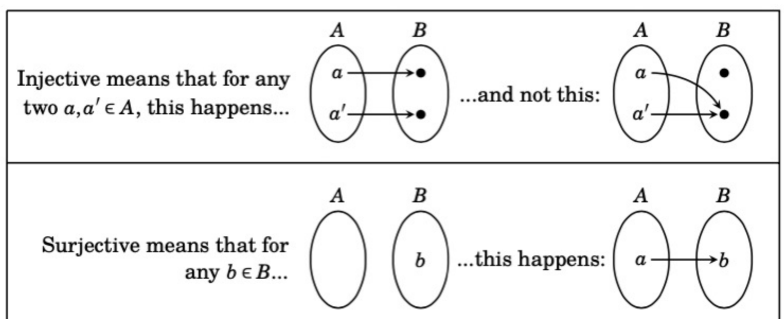
equivalent, if $\forall a, a' \in A$, if $a \neq a'$ then $f(a) \neq f(a')$.

(2) <u>surjective</u> (or <u>onto</u>) if $\forall b \in B$, $\exists a \in A$ s.t. $f(a) = b$,

equivalently, if $B = \mathrm{im}(f)$.

(3) <u>bijective</u> if it is both injective and surjective.

In pictures:

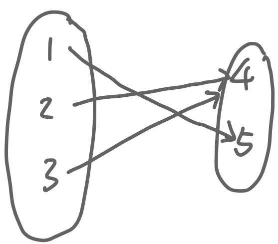


Examples:

- The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ $\forall x \in \mathbb{R}$ is neither inj. nor surj.:
  - $-1 \neq 1$ yet $f(-1) = f(1) = 1$, so f is not inj.
  - the number -1 is in the codomain $\mathbb{R}$ but not equal to x^2 for any $x \in \mathbb{R}$ (since $x^2 \geq 0$ $\forall x \in \mathbb{R}$), so f is not surj.

    (We noted that $\text{im}(f) = \mathbb{R}_{\geq 0} \neq \mathbb{R}$, so f is not surj.)

- $f: A \to B$



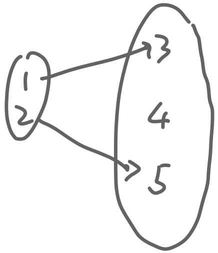
is not injective and is surj.

since $f(2) = f(3)$

or, by the pigeonhole principle!

Indeed, if $|A| = m$, $|B| = n$, $m, n < \infty$ and $m > n$, then no function from A to B is inj.

- $f : A \to B$



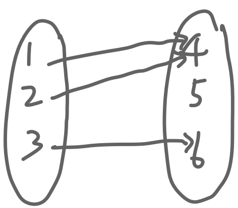
is inj but not surj.

$|A| = m$, $|B| = n$. $m, n < \infty$

$m < n$

$\Bigg\} \Rightarrow$ no function from A to B can be surj.

- $f : A \to B$



is neither inj nor surj.

$f(1) = f(2)$ $\qquad$ $5 \in B \setminus \mathrm{im}(f)$.

- $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ w/ $f(x) = \frac{1}{x} + 1 \quad \forall x \in \mathbb{R} \setminus \{0\}$.

Surj? No. since $1 \in \mathbb{R}$ but $1 \neq \frac{1}{x} + 1$ for any $x \in \mathbb{R} \setminus \{0\}$.

never 0

Inj? Yes. since $\frac{1}{x_1} + 1 = \frac{1}{x_2} + 1 \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in \mathbb{R} \setminus \{0\}$.