Math 2001. Lecture 2.

06.03.2022.

Last time:

- definitions and notation for sets
- containment and equality of sets, subset
- "elt in set" ($\in$) vs. "set contained in set" ($\subseteq$)
- Cartesian products and power sets of sets, and their cardinalities

Today:

- finishing the proof that $|P(A)| = 2^{|A|}$
- set operations: union, intersections, complement
- Venn diagrams
- proofs of set equalities; De Morgan's Law
- Indexed sets.

0. $|\mathcal{P}(A)| = 2^{|A|}$

Prop: If A is a finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

Key idea for the proof / technique / counting : think of dynamic processes.

earlier examples: A: entree options, B: drink options

To count $A \times B$ (entree-drink combos (e, d)),

think of what information you need to specify.

Step 1: specify the entree $\to$ $|A|$ choices

Step 2: specify the drink $\to$ $|B|$ choices.

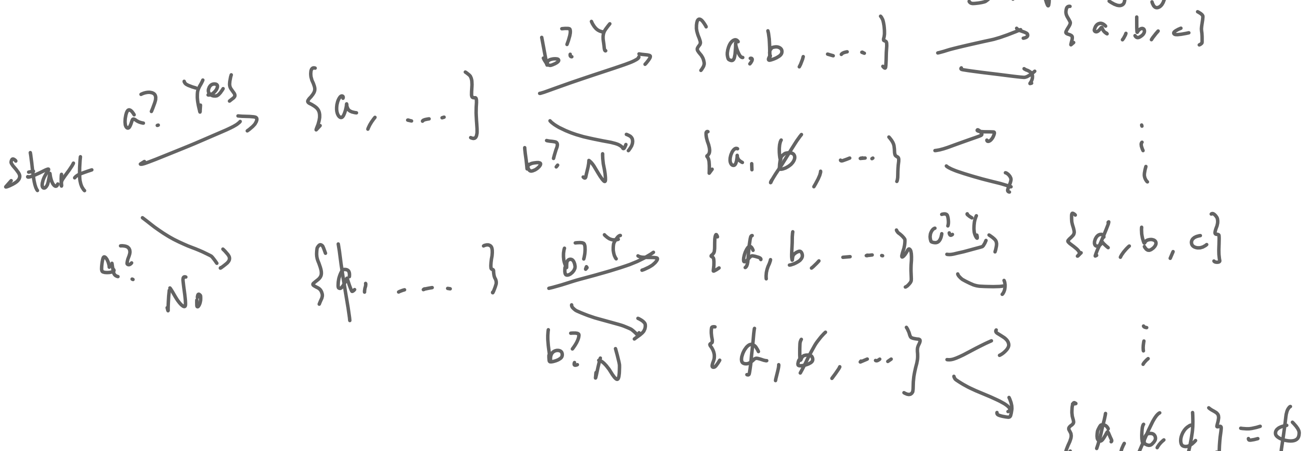
independent steps

Overall: $|A| \times |B|$ choices

Now, to count $\mathcal{P}(A)$, we should think about specify an elt of $\mathcal{P}(A)$, i.e., a subset of A, in suitable steps.

Pf: To specify a subset of A is to determine whether each elt of A should be included in the subset or not. We have 2 choices for each step corresponding to each elt in A, so there are $\underbrace{2\times 2\times 2\cdots\times 2}_{|A| \text{ copies}} = 2^{|A|}$ possible subsets of A. $\square$

E.g. $A=\{a,b,c\}$, $|A|=3$. Consider constructing/specifying a subset of A.



1. Set operations (new sets from old)

Def. Let A, B be two sets.

- The union of A and B is the set $A \cup B := \{x : x \in A \text{ or } x \in B\}$.
- The intersection of A and B is the set $A \cap B := \{x : x \in A \text{ and } x \in B\}$.
- The difference of A and B is the set $A - B = \{x : x \in A \text{ and } x \notin B\}$.

"$A \setminus B$" is another possible notation.

E.g. Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 7\}$, $C = \{2, 6, 7, 9\}$.

Then $A \cup B = \{1, 2, 3, 4, 5, 7\} = B \cup A$

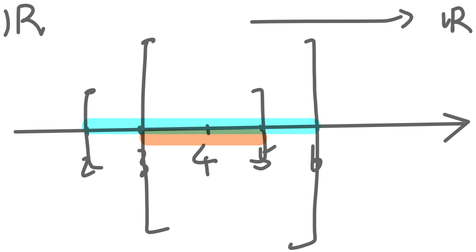
$A \cap B = \{3, 4\} = B \cap A$, $A \cap C = \{2\} = C \cap A$.

$A - B = \{1, 2\}$, $B - A = \{5, 7\}$. ($A - B \neq B - A$) "order matters"

$A - (B \cup C) = A - \{2, 3, 4, 5, 6, 7, 9\} = \{1\}$

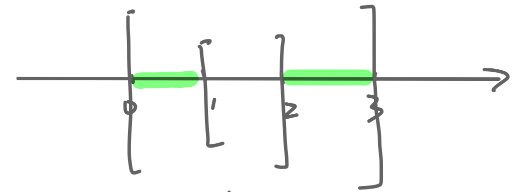
E.g. In the usual interval notation for $\mathbb{R}$

$[2,5] \cup [3,6] = [2,6]$

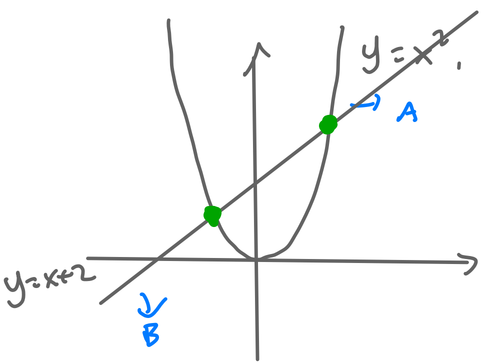


$[2,5] \cap [3,6] = [3,5]$

$[0,3] - [1,2] = [0,1) \cup (2,3]$



E.g. Consider the sets $A = \{(x, x^2) \mid x \in \mathbb{R}\} = \{(x,y) \mid x \in \mathbb{R}, y = x^2\} \subseteq \mathbb{R}^2$



$B = \{(x, x+2) \mid x \in \mathbb{R}\} = \{(x,y) \mid x \in \mathbb{R}, y = x+2\} \subseteq \mathbb{R}^2$.

Then $A \cap B = \left\{ (x,y) \,\middle|\, x,y \in \mathbb{R}, \underbrace{\begin{cases} y = x^2 \\ y = x+2 \end{cases}}_{*} \right\} = \{(-1,1), (2,4)\}$

$\rightarrow$ finding $A \cap B$ is to find the intersection of the curves/lines, which is to solve the equation system $(*)$.

One more operation: taking complements

Universe: when talking about sets of objects, we often have in mind a <u>universal set</u> or a <u>universe</u> U of "all (relevant) objects".

Sometimes we need to specify U; sometimes it's clear from context what U is.

Def: The <u>complement</u> of a set A (in a universe U) is the set $\bar{A} = \{x : x \notin A\}$ $(= \{x \in U : x \notin A\})$.

$= A^c$

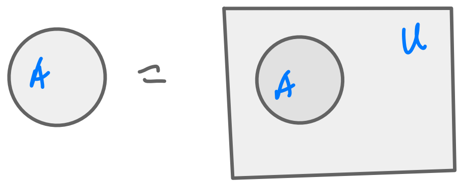
E.g. · If $U = \mathbb{Z}$ and A is the set of all even integers, then $\bar{A}$ is the set of all odd integers.

· If $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{2, 3, 5\}$, then $\bar{A} = \{1, 4, 6\}$.

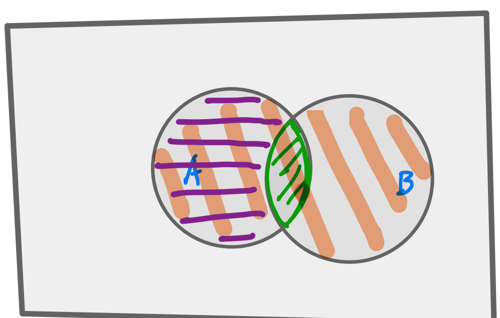
2. Venn diagrams

Venn diagrams are diagrams we often use to depict the interactions of / possible operations on a number of sets.

One set :



Two sets :

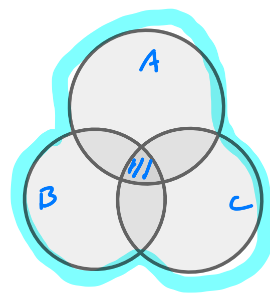


/// : $A \cap B$

\\\ : $A \cup B$

≡ : $A - B$ $\left(\begin{aligned} &= A - (A \cap B) \\ &= A \cap \overline{B} \end{aligned} \right)$

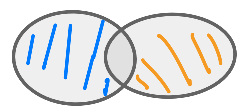
Three sets :



/// : $A \cap B \cap C$.

Venn diagrams can help us understand set equalities involving various set operations.

Examples:

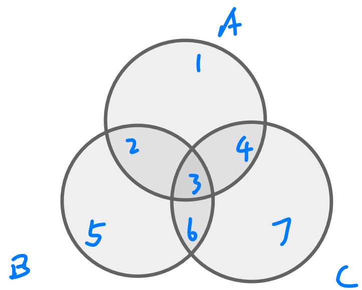
- $A - B \neq B - A$:  "///" $\neq$ "\\\"

- $(A \cup B) \cup C = A \cup (B \cup C)$: both are "all of A, B, C" ( ).

- Is it true that for any three sets A, B, C we always have

  $A \cap (B \cup C) = (A \cap B) \cup C$?

$\underbrace{A \cap (B \cup C)}_{X}$ vs. $\underbrace{(A \cap B) \cup C}_{Y}$



$A \cap (B \cup C)$ = Regions 2, 3, 4.

$(A \cap B) \cup C$ = Regions 2, 3, 4, 6, 7.

So, the two sets are not equal in general $\longrightarrow$ they are not equal whenever Regions 6 or 7 contain some elts.

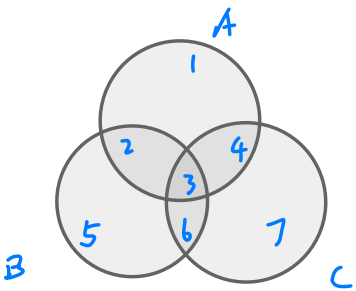
A specific counterexample: Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6\}$ and $C = \{3, 4, 6, 7\}$. Then

$A \cap (B \cup C) = \{2, 3, 4\}$ while

$(A \cap B) \cup C = \{2, 3, 4, 6, 7\}$.

Ex: Prove that $X \subseteq Y$ for any sets A, B, C.

· One more example : $A \cap (B \cup C) \overset{?}{=} (A \cap B) \cup (A \cap C)$



Analysis: LHS: $B \cup C$: 2–7

$A \cap (B \cup C)$: 2, 3, 4

RHS: $A \cap B$: 2, 3

$A \cap C$: 3, 4

$(A \cap B) \cup (A \cap C)$: 2, 3, 4.

So yes. LHS = RHS for all sets A, B, C.

Pf of the equality: We need to prove LHS $\subseteq$ RHS and RHS $\subseteq$ LHS.

LHS $\subseteq$ RHS : Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, we have $x \in B$ or $x \in C$. Now, if $x \in B$, then since $x \in A$, we have $x \in (A \cap B)$; if $x \in C$, then since $x \in A$, we have $x \in (A \cap C)$. So $x \in (A \cap B) \cup (A \cap C)$.

Therefore LHS $\subseteq$ RHS.

RHS $\subseteq$ LHS: Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$.

If $x \in A \cap B$, then $x \in A$ and $\underline{x \in B}$, so also $\underline{x \in B \cup C}$, so $x \in A \cap (B \cup C)$;

If $x \in A \cap C$, then $x \in A$ and $x \in C$, so also $x \in B \cup C$, so $x \in A \cap (B \cup C)$.

So $x \in A \cap (B \cup C)$. Therefore RHS $\subseteq$ LHS.

De Morgan's Laws:

Prop: Let A, B be two sets in a universe U. Then

(i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$ (ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Intuitively, (i): to fail to be in (either A or B) is to be in neither A nor B.

(ii): to fail to be in (both A and B) is to "miss" A or "miss" B.

Ex: Prove the prop. carefully.