Math 2001. Lecture 19. Final Exam: Friday, July 1. 06. 27. 2022.

Last time: · graphs and trees, and a related inductive proof

Today: More on (strong) mathematical induction:

- the fundamental thm of arithmetic

- Fibonnaci numbers

1. The fundamental thm of arithmetic

Thm: Every positive integer $n > 1$ has a unique factorization into prime numbers (up to reordering of the factors).

e.g. $60 = 2 \times 2 \times 3 \times 5 = 5 \times 3 \times 2 \times 2$

Pf: We first prove the existence of such a prime factorization by strong induction on n.

(1) Base case: $n = 2$. The number 2 is a prime. So $2 = 2$ is a prime decomp. of 2.

(2) Inductive step: Suppose that we have proven that $2, 3, 4, \cdots, k$ all have prime decomp. for some $k \in \mathbb{Z}_{\geq 1}$. We want to prove that

$k+1$ also has a prime decomp.

- If $k+1$ is itself a prime, then $k+1 = k+1$ is a prime decomp. of $k+1$.

- If $k+1$ is not prime, then $k+1$ has divisors a, b s.t. $1 < a, b < k+1$ and $k+1 = ab$. By the strong inductive hyp., there exist prime decomp. $a = p_1 p_2 \cdots p_m$ and $b = p_1' p_2' \cdots p_r'$.

  So $k+1 = ab = p_1 p_2 \cdots p_m \cdot p_1' p_2' \cdots p_r'$ is a prime decomp of $k+1$.

By (1) and (2), every int. $n > 1$ has a prime decomp.

To prove the uniqueness of the prime factorization of n, we again use strong induction.

(a) Base case: $n=2$. It's clear that $2=2$ is the only prime decomp. of 2.

(b) Inductive step (combined with "proof by contradiction"): Suppose, for contradiction, that some number in $\mathbb{Z}_{>1}$ does not have a unique factorization. Then there is a minimal such number n having at least two prime factorizations

$$n = a_1 a_2 \cdots a_m = p_1 p_2 \cdots p_r .$$

Since $p_1 \mid n = a_1 a_2 \cdots a_m$, it follows (from properties of prime numbers) that $p_1 \mid a_i$ for some $1 \le i \le l$. Without loss of generality, we may assume $p_1 \mid a_1$.

whence $p_1 = a_1$.

But then we have

$$n = (a_1) a_2 \cdots a_\ell = p_1 p_2 \cdots p_r = (a_1) p_2 \cdots p_r ,$$

so we have $a_2 a_3 \cdots a_\ell = p_2 \cdots p_r$ are two distinct prime decompositions of the number $n/a_1 = n/p_1 < n$. This contradicts the assumption that n is the smallest pos. int. with distinct prime decompositions, so it must be the case that every pos. int greater than 1 has a unique prime decomp. $\square$

2. Fibonacci numbers

Def: The Fibonacci sequence is the recursively defined sequence $F_1, F_2, F_3, \ldots\ldots$ given by the initial values $F_1 = 1, F_2 = 1$ and the recursion $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.

e.g. $F_3 = F_1 + F_2 = 1+1 = 2$; $F_4 = F_2 + F_3 = 1+2 = 3$;

$F_5 = F_3 + F_4 = 2+3 = 5$; $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

The numbers in the Fibonacci sequence are called the Fibonacci numbers.

Rmk: The recursive nature of the Fibonacci sequence allows inductive proofs for many properties of the Fibonacci sequence / numbers.

Prop 1. The Fib. numbers satisfy $F_{n+1}^2 - F_{n+1}F_n - F_n^2 \overset{(*)}{=} (-1)^n \ \forall n \geq 1$.

Pf: We prove (*) by induction on n.

Base case(s): $n=1$. $F_2^2 - F_2F_1 - F_1^2 = 1^2 - 1\cdot 1 - 1^2 = -1 = (-1)^1$,

so (*) holds.

$n=2$. $F_3^2 - F_3F_2 - F_2^2 = 2^2 - 2\cdot 1 - 1^2 = 1 = (-1)^2$,

so (*) holds.

Inductive step: Suppose (*) holds for $n \in \{1, 2, \ldots, k\}$ for some $k \in \mathbb{Z}_{\geq 2}$.

We need to prove (*) for the case $n = k+1$ (≥ 3).

By the def. of the Fib. sequence, we have, in this case,

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (F_n + F_{n-1})^2 - (F_n + F_{n-1})F_n - F_n^2$$

$$= \cancel{F_n^2} + 2F_nF_{n-1} + F_{n-1}^2 - F_n^2 - F_{n-1}F_n - F_n^2$$

$$= -F_n^2 + F_nF_{n-1} + F_{n-1}^2$$

The last expression equals $-\left(\underline{F_n^2 - F_nF_{n-1} - F_{n-1}^2}\right)$

where by induction $\underline{F_n^2 - F_nF_{n-1} - F_{n-1}^2} = \underline{(-1)^n}$,

so we have $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)\underline{(-1)}^n = (-1)^{n+1}$.

i.e., (*) holds for $n = k+1$, as desired.

By (1) and (2), we conclude that (*) holds for all $n \geq 1$. $\square$

<u>Rmk</u>: Since the Fib. sequence is defined using a <u>2nd</u> order recursion, in inductive proofs it's often useful to check the base cases for both F_1 and F_2 (rather than just one base case for F_1).

Prop 2. $\forall n \in \mathbb{Z}_{\geq 1}$, we have $\displaystyle\sum_{i=1}^{n} F_i^2 \overset{(*)}{=} F_n F_{n+1}$.

Pf: We use induction.

Base cases: $n=1$, $F_1^2 = 1$, $F_1 F_2 = 1 \cdot 1 = 1$,

so $(*)$ holds.

$n=2$, $F_1^2 + F_2^2 = 1+1 = 2$, $F_2 \cdot F_3 = 1 \cdot 2 = 2$,

so $(*)$ holds.

Inductive proof (with a slightly different template): Suppose $N \geq 3$ and suppose that we have proven $(*)$ for all $n \in \{1, 2, \ldots, N-1\}$. We hope to prove $\displaystyle\sum_{i=1}^{N} F_i^2 = F_N F_{N+1}$. To do so, note that

$$\sum_{i=1}^{N} F_i^2 = \left(\sum_{i=1}^{N-1} F_i^2\right) + F_N^2 \overset{\text{induction}}{=\!=} \left(F_{N-1} F_{N-1+1}\right) + F_N^2 = F_{N-1} F_N + F_N^2 = F_N\left(F_{N-1} + F_N\right)$$
$$= F_N F_{N+1}.$$

Thus (*) holds for the case $n = N$.

It follows that (*) holds for all $n \in \mathbb{Z}_{\geq 1}$. $\square$

A consequence of Prop 1. $\left(F_{n+1}^2 - F_{n+1}F_n - F_n^2 \overset{(*)}{=} (-1)^n \quad \forall\, n \geq 1. \right)$

Facts: (i) $F_n > 0 \quad \forall n \geq 1$, and $F_{n+1} > F_n \quad \forall n > 2$.

$$(*) \Rightarrow \frac{F_{n+1}^2}{F_n^2} - \frac{F_{n+1}F_n}{F_n^2} - \frac{F_n^2}{F_n^2} = \frac{(-1)^n}{F_n^2},$$

i.e.,
$$\left(\frac{F_{n+1}}{F_n}\right)^2 - \left(\frac{F_{n+1}}{F_n}\right) - 1 = \frac{(-1)^n}{F_n^2}.$$

(ii) Let $r_n = \frac{F_{n+1}}{F_n}$. Then $\lim_{n\to\infty} r_n$ exists.

Q: What is $r := \lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$?

A: By (i) and (ii), we have $r \geq 0$ and

$$\lim_{n\to\infty}\left(r_n^2 - r_n - 1 \right) = \lim_{n\to\infty}\left(\frac{(-1)^n}{F_n^2}\right)$$

$$r^2 - r - 1 = 0,$$

so $r \in \left\{ \frac{1 \pm \sqrt{1+4}}{2} \right\} = \left\{ \frac{1 \pm \sqrt{5}}{2} \right\}$, and in fact $\frac{1+\sqrt{5}}{2}$.

"the golden ratio".

Φ

One more inductive proof: "Entries the diagonals of Pascal's triangle sum to Fibonacci numbers."

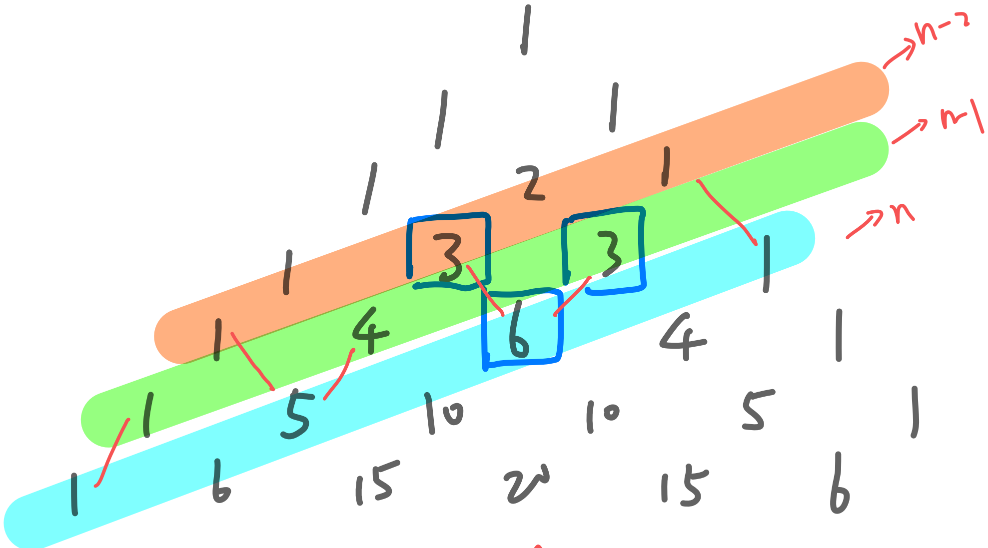
Q1: How do we make the statement precise?

$$
\begin{array}{c}
1\\
1\quad 1\\
1\quad 2\quad 1\\
1\quad \boxed{3}\quad \boxed{3}\quad 1\\
1\quad 4\quad \boxed{6}\quad 4\quad 1\\
1\quad 5\quad 10\quad 10\quad 5\quad 1\\
1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1
\end{array}
$$

- $\binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5 = F_5$
- $\binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 8 = F_6$
- $\binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 13 = F_7$

A: The precise statement is $\forall n \geq 1$.

S_n: $\begin{cases} \text{if } n \text{ is even, say } n = 2k, \text{ then } \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{k}{k} = F_{n+1} \\ \text{if } n \text{ is odd, say } n = 2k+1, \text{ then } \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{k+1}{k} = F_{n+1}. \end{cases}$



Q: How do we prove the statement?

A: Use strong induction and the fact that $\binom{n}{k} \overset{(*)}{=} \binom{n-1}{k} + \binom{n-1}{k-1}$.

a representative proof: Sum of the entries in the nth diagonal $\overset{(*)}{=}$ the same sum for the $(n-1)$th diag + the same sum for the $(n-2)$th diagonal

$\underset{\text{induction}}{=} F_n + F_{n-1} = F_{n+1}$. ✓