Math 2001. Lecture 18.

06. 24. 2022.

Last time: · induction and strong induction:

To prove S_n (statements depending on n) $\forall n \in \mathbb{Z}_{\geq 0}$,

it suffices to

(1) prove the first few S_i's.

(2) prove that either $S_k \Rightarrow S_{k+1}$ $\forall k \in \mathbb{Z}_{\geq 0}$

or that $S_0 S_1 \wedge S_2 \wedge \cdots \wedge S_k \Rightarrow S_{k+1}$. (strong ind.)

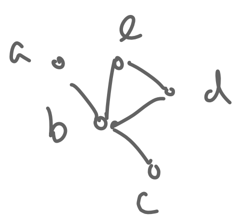
Today: · more strong induction proofs

1. Proofs using strong induction

A prop. about "trees". "$|E| = |V| - 1$ for trees".

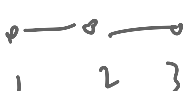
Background: An (undirected) graph, abstractly, is the data of a set V of "vertices" and a set E of "(undirected) edges" where each edge contains two vertices (so $E \subseteq V \times V$).

Eg.



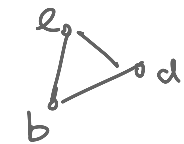
$\longrightarrow$ $V = \{a, b, c, d, e\}$

$E = \{\{a,b\}, \{b,c\}, \{b,d\}, \{b,e\}, \{d,e\}\}$.



$\longleftarrow$ $V = \{1, 2, 3\}$, $E = \{\{1,2\}, \{2,3\}\}$

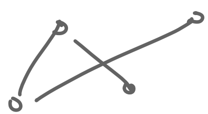
- A *cycle* in a graph $G=(V,E)$ is a sequence $v_1, v_2, \cdots, v_k$ of vertices in V s.t. $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}$ are all edges in E. (e.g. b-d-e-b gives a cycle

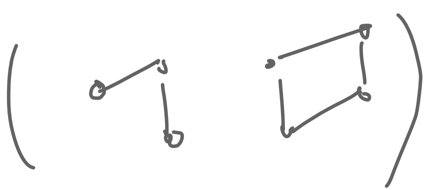


in our first example).

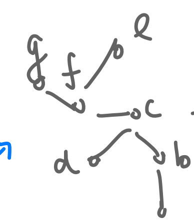
- A graph $G=(V,E)$ is connected if there is a path from x to y $\forall x,y \in V$.

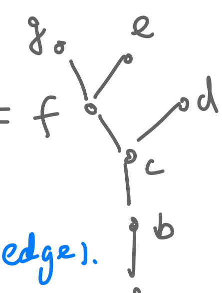
a sequence $x = v_1, v_2, \cdots, v_k = y$ s.t. $\{v_i, v_{i+1}\} \in E$ $\forall\, 1 \le i \le k-1$

(1)  is connected.

(2)  is not connected.

e.g.



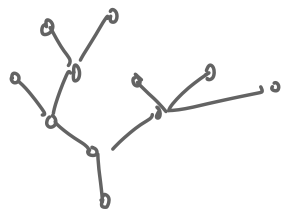
= 

Def: A *tree* is a connected graph with no cycle.
"acyclic"

7 vert. 6 edges.

Prop: If a tree has n vertices for some $n \in \mathbb{Z}_{\geq 1}$, then it has $n-1$ edges. $\quad S_n: |E| = |V| - 1$ if $|V| = n$.

e.g.



$\rightarrow \quad |V| = 11, \quad |E| = 10.$

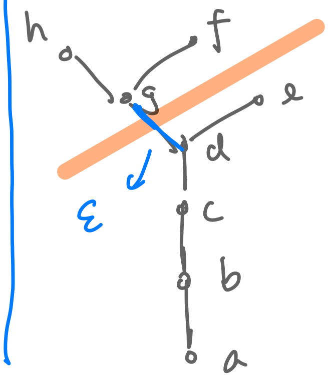
Pf (sketch): We prove the prop. by strong induction on n.

Base case: Consider $n=1$. A tree with one vertex must have no edge, i.e., zero edges. so S_1 holds.

(strong) Inductive step: Suppose $S_1, S_2, \ldots, S_k$ hold for some $k \in \mathbb{Z}_{\geq 1}$. We want to show that S_{k+1} holds. that is, we need to show that every tree with $k+1$ vertices must have k edges.

Suppose $T=(V,E)$ is a tree with $(k+1)$ vertices.

Pick any edge $\varepsilon \in E$ of T and remove it (but do not remove the vertices it connects).



Doing so results in two smaller trees T_x, T_y with x vertices and y vertices, respectively, for some positive integers x, y s.t. $x+y=k+1$.

By the strong inductive hypothesis, S_x and S_y hold, so T_x has $x-1$ edges and T_y has $y-1$ edges. Thus, the original tree T has

$$\underbrace{(x-1)}_{\text{edges in } T_x} + \underbrace{(y-1)}_{\text{edges in } T_y} + \underbrace{1}_{\text{the removed edge}} = x+y-1 = (k+1)-1 = k,$$

as desired. □