Math 2001. Lecture 17. 06.23.2022.

Last time:

- Proofs of set containments / equalities
- disproofs and T/F problems

Today:

- Mathematical induction
- strong mathematical induction

$\Big\}$ Ch. 11.

1. What is mathematical induction?

- The (type of) problem: prove a statement S_n (depends on n) for all $n \geq 0$ (or $\forall n \geq 1$, $\forall n \geq 2$, ...).

  E.g. Show that for every $n \in \mathbb{Z}_{\geq 0}$, we have

$$\underbrace{1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}}_{S_n}.$$

- The proof strategy: to prove "S_n holds for all n" in two steps:

  (1) the "base case": prove S_n for the smallest relevant value of n. "first statement"

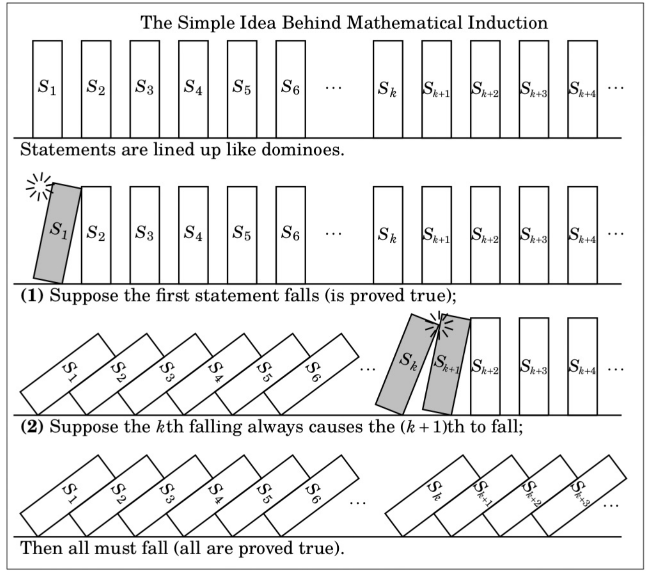
  (2) the "inductive step": prove that for all k ($\in \mathbb{Z}_{\geq 0}$), if S_k holds then S_{k+1} holds.

  "inductive hypothesis"

  "chain reaction" / "domino effect"

It follows that S_n holds for all n.

- The intuitive idea:



- The key in a typical proof by (mathematical) induction: to help prove the inductive step. study how the quantities / objects / expressions in S_k relate to those in S_{k+1}. → find the recursion.

2. Examples

1. $\forall n \in \mathbb{Z}_{>0}$, $\underbrace{1+2+3+\cdots+n = \frac{n(n+1)}{2}}_{S_n}$.

Pf: We use mathematical induction (on n).

(1) The base case: $n=1$. S_1 just says that $1 = \frac{1\cdot(1+1)}{2}$, i.e., $1 = \frac{1\cdot 2}{2}$, which is certainly true.

(2) The inductive step: We need to show that $S_k \Rightarrow S_{k+1}$ $\forall k \in \mathbb{Z}_{>0}$. So suppose S_k $(S_k \Rightarrow S_{k+1})$ holds, i.e., $1+2+\cdots+k = \frac{k(k+1)}{2}$.

It follows that $1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2}+(k+1) = \frac{k(k+1)}{2}+\frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

So S_{k+1} holds.

By (1) and (2), it follows by the principle of math. induction that S_n holds for all $n \in \mathbb{Z}_{>0}$. $\square$

2. If $n \in \mathbb{Z}_{>0}$, then $\underbrace{1+3+5+7+\cdots+(2n-1) = n^2}_{S_n}$.

e.g. $1 = 1^2$, $1+3=4=2^2$, $1+3+5=9=3^2$, ...

Pf: We prove the equality S_n by induction.

(1) Base case: $n=1$. S_1 says $1 = 1^2$, which is clearly true.

(2) Inductive step: Suppose S_k holds for some $k \in \mathbb{Z}_{>0}$. Then

($S_k \Rightarrow S_{k+1}$)

$$1+3+\cdots+(2k-1) \overset{(i)}{=} k^2.$$

We need to show S_{k+1} holds, i.e., $1+3+\cdots+(2k-1)+(2(k+1)-1) \overset{(ii)}{=} (k+1)^2$.

To prove (ii), note that

$$1+3+\cdots+(2k-1)+(2(k+1)-1) = k^2+(2k+1) = (k+1)^2.$$

By (1) and (2), it follows that S_n holds for all $n \in \mathbb{Z}_{>0}$. $\square$

3. If $n \in \mathbb{Z}_{\geq 0}$, then $5 \mid (n^5 - n)$.

e.g. $0^5 - 0 = 0$, $1^5 - 1 = 1 - 1 = 0$, $2^5 - 2 = 32 - 2 = 30$, $3^5 - 3 = 243 - 3 = 240$. ✓

Pf: We prove the statement $S_n : 5 \mid n^5 - n$ $\forall n \in \mathbb{Z}_{\geq 0}$ by induction.

(1) Base case: $n = 0$. Then $n^5 - n = 0^5 - 0 = 0 - 0 = 0$, which is indeed divisible by 5, so S_0 holds.

(2) Inductive step: Suppose S_k holds for some $k \in \mathbb{Z}_{\geq 0}$, i.e., suppose that $5 \mid k^5 - k$. We hope to show that $5 \mid (k+1)^5 - (k+1)$. Note that

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + \cancel{1} - (k + \cancel{1}) \quad \left(\text{by the binomial thm}\right)$$

$$= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k$$

$$= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$$

Here, $5(k^4+2k^3+2k^2+k)$ is divisible by 5,

as is k^5-k by the inductive hypothesis,

So it follows that $5 \mid (k+1)^5-(k+1)$, i.e., S_{k+1} holds.

By (1) and (2), it follows that $5 \mid n^5-n \;\; \forall n \in \mathbb{Z}_{\geq 0}$. $\square$

4. $\forall n \in \mathbb{Z}_{\geq 0}.\quad \sum_{i=0}^{n} i \cdot i! \overset{S_n}{=} (n+1)!-1.$

$\downarrow$: the recursion

pf: E.X.

e.g. $n=0$. S_0 : $\underline{0 \cdot 0!} \overset{?}{=} 1!-1$, $0 \cdot 1 \overset{?}{=} 1-1$. ✓

$n=1$. S_1 : $\underline{\underline{0 \cdot 0!} + 1 \cdot 1!} \overset{?}{=} 2!-1$, $0+1 \overset{?}{=} 2-1$ ✓

$n=2$ S_2 : $\underline{\underline{0 \cdot 0! + 1 \cdot 1!} + 2 \cdot 2!} \overset{?}{=} 3!-1$, $1+4 \overset{?}{=} 6-1$. ✓

$n=3$ S_3 : $\underline{\quad \cdots \quad} + 3 \cdot 3! \overset{?}{=} \cdots$

5. $\forall n \in \mathbb{Z}_{\geq 1}$, $2^n \leq 2^{n+1} - 2^{n-1} - 1$.

e.g. $n=1$. want $2^1 \leq 2^2 - 2^0 - 1$, i.e., $2 \leq \underbrace{4-1-1}_{2}$. ✓

$n=2$. want $2^2 \leq 2^3 - 2^1 - 1$, i.e., $4 \leq \underbrace{8-2-1}_{5}$ ✓

$n=3$. want $2^3 \leq 2^4 - 2^2 - 1$, i.e., $8 \leq \underbrace{16-4-1}_{11}$ ✓

Pf: We prove the inequality S_n : $2^n \leq 2^{n+1} - 2^{n-1} - 1$ by induction.

(1) Base case: when $n=1$, we have $2^n = 2^1 = 2$ and $2^{n+1} - 2^{n-1} - 1 = 4 - 1 - 1 = 2$,

so S_1 holds (since $2 \leq 2$).

(2) Inductive step: Suppose S_k holds for some $k \in \mathbb{Z}_{\geq 1}$, i.e., suppose $2^k \leq 2^{k+1} - 2^{k-1} - 1$.

Then, $2^{k+1} = 2 \cdot 2^k \leq 2 \cdot (2^{k+1} - 2^{k-1} - 1) = 2^{k+2} - 2^k - 2 < 2^{k+2} - 2^k - 1$

$= 2^{(k+1)+1} - 2^{(k+1)-1} - 1$

It follows that S_{k+1} holds.

By (1) and (2), it follows that S_n holds for all $n \in \mathbb{Z}_{\geq 1}$.

6. If $n \in \mathbb{Z}_{\geq 1}$, then $(1+x)^n \geq 1+nx$ for all $x \in \mathbb{R}$ with $x > -1$.

E.g. $n=1$: want $(1+x)^1 \geq 1+x$? ✓

$n=2$: want $(1+x)^2 \geq 1+2\cdot x$? $\quad 1+2x+x^2 \geq 1+2x$?

✓, since $x^2 \geq 0$.

$n=3$: want $(1+x)^3 \geq 1+3\cdot x$? $\quad 1+3x+3x^2+x^3 \geq 1+3x$?

$3x^2+x^3 \geq 0$?

More importantly, where's the recursion ? $\quad 3x^2+x^3 = x^2(x+3)$, true since $x > -1$.

Pf: Suppose $x > -1$. We prove that the inequality $S_n : (1+x)^n \geq 1+nx$ holds for all $n \in \mathbb{Z}_{\geq 1}$ by induction on n.

(1) Base case : when $n=1$, we have $(1+x)^n = 1+x \geq 1+1\cdot x$, so S_1 holds.

(2) Inductive step : Suppose S_k holds for some $k \in \mathbb{Z}_{\geq 1}$, i.e. that $(1+x)^k \geq 1+kx$.

We need to show that $(1+x)^{k+1} \geq 1+(k+1)\cdot x$.

To do so, we note that

$$(1+x)^{k+1} = (1+x)^k \cdot (1+x)$$

$$\geq (1+kx) \cdot (1+x) \quad \text{by the ind. hyp.}$$

$$= 1 + (k+1)x + kx^2.$$

$$\geq 1 + (k+1)x \quad \text{since } kx^2 \geq 0.$$

It follows that S_{k+1} holds.

By (1) and (2), it follows that S_n holds $\forall n \in \mathbb{Z}_{\geq 1}$, as desired. $\square$

2. Strong mathematical induction

Goal: same as before $\longrightarrow$ to prove that a statement S_n (depending on n) holds for all n.

The proof outline: 1. Base case, or base cases: prove that the first few S_i's are true.

2. the strong inductive step: prove that for every k, we have $\underbrace{S_1 \wedge S_2 \wedge \cdots \wedge S_k}_{\text{"and"}} \Longrightarrow S_{k+1}$.

Difference from basic induction: in the inductive step, we use not only S_k as a hypothesis but instead all of $S_1, S_2, \cdots, S_k$.

However, the key idea is still to find and take advantage of suitable recursions

Example.

Prop: Any postage of 8 cents or more can be formed by combining 3-cent and 5-cent stamps.

Analysis:

base case : 8 . $\quad 8 = 3 + 5$

next few : 9 . $\quad 9 = 3 + 3 + 3$

10 . $\quad 10 = 5 + 5$

11 . $\quad 11 = 3 + 3 + 5$

12 . $\quad 12 = 3 + 3 + 3 + 3$

$13 = 3 + 5 + 5$

⋮

Pf: It suffices to prove to prove the claim S_n: n is a sum of 3s and 5s

for all $n \in \mathbb{Z}_{\geq 8}$. We do so by strong induction on n.

(1) Base cases: for $n = 8, 9$ or 10, we have $8 = 3+5$, $9 = 3+3+3$ and

$10 = 5+5$, so S_n holds.

(2) Strong ind. step: suppose $S_8, S_9, S_{10}, \ldots, S_k$ all hold for some $k \geq 10$.

We need to show that S_{k+1} holds, ie, that $k+1$ is a sum of 3s and 5s.

Now, $k+1 = 3 + (k-2)$ and $k-2$ is a sum of 3s and 5s by the

strong ind. hyp, so $k+1$ must also be a sum of 3s and 5s (by S_{k-2}).

So S_{k+1} holds.

By (1) and (2), it follows that S_n holds for all $n \in \mathbb{Z}_{\geq 8}$. $\square$