Math 2001. Lecture 16.

06. 22. 2022.

Last time:

- existence proofs
  - constructive or nonconstructive

Today:

- proofs involving sets
  - set containment ($\subseteq$) or equality ($=$)

Recall:

- If A, B are sets, then $A \subseteq B$ iff every elt in A is in B.
  "if $x \in A$ then $x \in B$ (for $x \in U$)"
  U: the universe
- Two sets A, B are equal iff $A \subseteq B$ and $B \subseteq A$.

1. Set containments and equalities

E.g. (8.5) Prove that $\{x \in \mathbb{Z} \mid 18 \mid x\} \subseteq \{x \in \mathbb{Z} \mid 6 \mid x\}$.

Pf: Let $y \in \{x \in \mathbb{Z} \mid 18 \mid x\}$. Then $18 \mid y$, so $y = 18k$ for some $k \in \mathbb{Z}$.

Thus, $y = 18k = (6 \cdot 3) \cdot k = 6 \cdot (3k)$ where $3k \in \mathbb{Z}$.

So $6 \mid y$, hence $y \in \{x \in \mathbb{Z} \mid 6 \mid x\}$, and we are done.

E.g. (8.7) Show that $\underbrace{\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 6\}}_{LHS} \subseteq \underbrace{\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 3\}}_{RHS}$.

Pf: Let $(x,y) \in LHS$. Then $x \equiv y \pmod 6$, so $6 \mid x-y$.

Since $3 \mid 6$, it follows that $3 \mid x-y$, so $x \equiv y \pmod 3$.

It follows that $(x,y) \in RHS$, so $LHS \subseteq RHS$. $\square$

(8.8) Prove that A, B are sets, then $P(A) \cup P(B) \subseteq P(A \cup B)$.

Pf: Take an (arbitrary) elt / set $X \in P(A) \cup P(B)$.

Then $X \in P(A)$ or $X \in P(B)$.

- if $X \in P(A)$, then $X \subseteq A$. Since $X \subseteq A$ and $A \subseteq A \cup B$, we have $X \subseteq A \cup B$.

- Similarly, if $X \in P(B)$ then we also have $X \subseteq A \cup B$.

So in all cases we have $X \subseteq A \cup B$, so $X \in P(A \cup B)$.

Therefore $P(A) \cup P(B) \subseteq P(A \cup B)$. $\square$

(8.9) Let A, B be sets. If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

Pf: Suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. We will show $A \subseteq B$ by showing that every elt $x \in A$ is also an elt in B.

Let $x \in A$. Then $\{x\} \subseteq A$, hence $\{x\} \in \mathcal{P}(A)$.

Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ by assumption, it follows that $\{x\} \in \mathcal{P}(B)$.

Thus, we have $\{x\} \subseteq B$. It follows that $x \in B$.

It follows that $A \subseteq B$, so we are done. $\square$

(8.10) Prove that $\underbrace{\{n \in \mathbb{Z} : 35 \mid n\}}_{A} = \underbrace{\{n \in \mathbb{Z} : 5 \mid n\} \cap \{n \in \mathbb{Z} : 7 \mid n\}}_{B}$.

Pf: Let A and B denote the sets on the left and right of the equality, respectively.

($A \subseteq B$) Suppose $k \in A$. Then $35 \mid k$

· Since $5 \mid 35$ and $35 \mid k$, we have $5 \mid n$, so $k \in \{n \in \mathbb{Z} : 5 \mid n\}$.

· similarly, since $7 \mid 35$, we have ...... $k \in \{n \in \mathbb{Z} : 7 \mid n\}$.

It follows that $k \in B$, hence $A \subseteq B$.

($B \subseteq A$): Suppose $k \in B$. Then $5 \mid k$ and $7 \mid k$. It follows that in the unique prime decomp. of k we have $k = 5 \cdot 7 \cdot \underbrace{\cdots\cdots}_{\text{(other primes}} = 35 \cdot \cdots\cdots$,

so $35 \mid k$. Thus, we have $k \in A$, so $B \subseteq A$.

We conclude that $A = B$, as desired. □

(8-11) Suppose A, B, C are sets, with $C \neq \phi$. Prove that if $A \times C = B \times C$,

then $A = B$.

they would be ϕ (and equal) if $C = \phi$.

Pf: Suppose $A \times C = B \times C$. We will prove $A = B$

by proving $A \subseteq B$ and $B \subseteq A$.

($A \subseteq B$) Let $a \in A$. Since $C \neq \phi$, we may pick an elt $c \in C$.

Then $(a, c) \in A \times C$. Since $A \times C = B \times C$, it follows that $(a, c) \in B \times C$.

It further follows that $a \in B$. Therefore $A \subseteq B$.

($B \subseteq A$) Similarly, we have $B \subseteq A$.

It follows that $A = B$.

(8.13.) Let A, B, C be sets. Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Pf 1: Do the $\subseteq, \supseteq$ proof.

Pf 2: Note that $LHS = \{(x,y) \mid x \in A \text{ and } y \in B \cap C\}$

$= \{(x,y) \mid x \in A \text{ and } y \in B \text{ and } y \in C\}$

$= \{(x,y) \mid (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\}$

$= \{(x,y) \mid x \in A \text{ and } y \in B\} \cap \{(x,y) \mid x \in A \text{ and } y \in C\}$

$= (A \times B) \cap (A \times C) = RHS.$ $\square$

2. Disproofs / True/False problems

To disprove a universal statement "$\forall x \in S,\ p(x)$", it suffices to find one counterexample, i.e., one example s.t. $p(x)$ does not hold.

E.g. · Statement: Every prime number is odd.

The statement is false because 2 is a counterexample: it is prime but not odd.

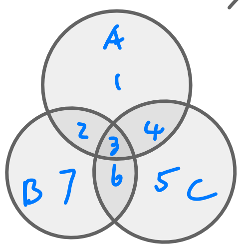
· Statement: if $n \in \mathbb{Z}$ then $\underbrace{n^2 - n + 11 \text{ is prime}}_{p(n)}$.

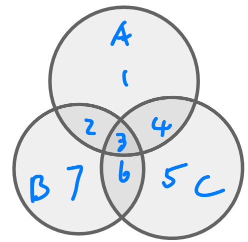
$\forall n \in \mathbb{Z}$

It's false since for $n = 11 \in \mathbb{Z}$ we have $n^2 - n + 11 = 11^2 - 11 + 11 = 11^2$, which is not prime.

Eg. T/F? "$\forall$ sets A, B, C, we have $A-(B\cap C) \overset{(*)}{\overset{?}{=}} (A-B)\cap(A-C)$?"

Analysis: As before, we study $(*)$ via Venn diagrams first.





$B\cap C$: 3, 6

$A-B\cap C$: 1, 2, 4.

$\left.\begin{array}{l} A-B : 1, 4 \\ A-C : 1, 2 \end{array}\right\} \Rightarrow (A-B)\cap(A-C) : 1$

$(*)$ should fail whenever there is an elt in region 2 or region 4.

Soln: The statement is false. Here's one counter-example: let $A=\{1,2,3,4\}$, $B=\{2,3,6,7\}$ and $C=\{3,4,5,6\}$. Then $A-(B\cap C)=\{1,2,3,4\}-\{3,6\}=\{1,2,4\}$ while $(A-B)\cap(A-C)=\{1,4\}\cap\{1,2\}=\{1\}$, so $A-(B\cap C)\neq(A-B)\cap(A-C)$. $\square$

To disprove an existential claim "$\exists x \in U, p(x)$", we need to show that $p(x)$ must fail for all $x \in U$ (so it's not enough to find one example x for which $p(x)$ fails).

E.g. "There is a real number $x \in \mathbb{R}$ s.t. $x^4 < x < x^2$."

The statement is false, since there is no real number $x \in \mathbb{R}$ w/
We'll prove this by contradiction.
$x^4 < x < x^2$: Let $x \in \mathbb{R}$. Suppose $x^4 < x$, then since $x^4 \geq 0$, we must have $x > 0$.
Now, since $x < x^2$ and $x > 0$, we have $1/x > 0$ and $x < x^2$. So

$$x \cdot \frac{1}{x} < x^2 \cdot \frac{1}{x}, \quad \text{i.e., } 1 < x.$$

But then $x^3 > 1$, hence $x^4 = x \cdot x^3 > x$, a contradiction.
It follows that we can't have both $x^4 < x$ and $x < x^2$. $\square$