# Math 2001. Lecture 11.

06.15.2022.

**Last time:**

- bars-and-stars problems
- multiset permutations (the word problem)

  e.g. # spellings of the letters in MISSISSIPPI

  $$= \frac{11!}{4!\,4!\,2!}$$

- summary of counting problems and techniques

**Today:**

- the pigeonhole principle
- the division principle

# 1. The pigeonhole principle.

**The principle:** Place $n$ objects into $k$ boxes where $n, k \in \mathbb{Z}_{>0}$.

(1) If $n > k$, then at least one box contains more than one object.

(2) If $n < k$, then at least one box contains no object.

**Pf:** (1) The conclusion holds because otherwise the boxes would contain at most $k$ and hence fewer than $n$ objects.

(2) - - - - - - - - - - - - -

- - at least $k$ and hence more than $n$ objects

$\square$

**Rmk:** The key to using the pigeonhole principle is often to design the appropriate "boxes".

**Examples:**

(1) Show that if six integers are chosen at random, then at least two of them have the same remainder when divided by 5.

e.g.

$$\begin{array}{cccccc} 1, & 3, & 7, & 12, & 25, & -3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 1\ \checkmark & 0 & 1 & 3 \end{array} \quad ; \quad \begin{array}{cccccc} 1, & 2, & 3, & 4, & 5, & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 0 & 1\ \checkmark \end{array}$$

"Boxes"?

Use the possible values of the remainder (when dividing by 5).

Pf: When dividing an integer by 5 there are 5 possible remainders, namely 0, 1, 2, 3, 4. Let $B_i = \{$the integers that have remainder $i$ when divided by $5\}$ for each $i \in \{0, 1, 2, 3, 4\}$.

Then any six chosen integers fall into one of the five sets (boxes) $B_0, B_1, B_2, B_3, B_4$. It follows from the pigeonhole principle that for some $j$, $B_j$ contains at least two of the six integers, in which case those two integers have the same remainder ($j$) when divided by 5. $\square$

(2). Pick six integers from the set $X = \{0, 1, 2, \cdots, 7, 8, 9\}$.

Show that two of them must add to 9. (*)

e.g. $\{2, 4, 5, 6, 8, 9\} \to 4+5 = 9$.

$\{0, 1, 2, 3, 4, 7\} \to 2+7 = 9$.

Note how we can design these "boxes" by thinking about "avoiding (*)".

Soln: Consider the five pairs ("boxes") $\underset{X_0}{\{0, 9\}}$, $\underset{X_1}{\{1, 8\}}$, $\underset{X_2}{\{2, 7\}}$, $\underset{X_3}{\{3, 6\}}$ and $\underset{X_4}{\{4, 5\}}$.

They partition $X$ and the two numbers in each pair add up to 9.

Now take any six of the elts in $X$. Then by the pigeonhole principle, two of the six picked numbers must appear in the same pair $X_i$ for some $i \in \{0, 1, 2, 3, 4\}$. Those two numbers add to 9.

## 2. The division principle.

**Def:** For any real number $r$,

the floor of $r$, denoted by $\lfloor r \rfloor$, is the largest int. $k$ s.t. $k \le r$.

e.g. $\lfloor 2 \rfloor = 2$, $\lfloor 2.1 \rfloor = 2$, $\lfloor \pi \rfloor = 3$, $\lfloor -2.1 \rfloor = -3$.

The division principle ↓

the ceiling of $r$, denoted by $\lceil r \rceil$, is the smallest int $k$. s.t. $k \ge r$.

e.g. $\lceil 2 \rceil = 2$, $\lceil 2.1 \rceil = 3$, $\lceil \pi \rceil = 4$, $\lceil -2.1 \rceil = -2$

**Prop:** Place $n$ objects into $k$ boxes $(k, n \in \mathbb{Z}_{>0})$.

1) At least one of the boxes gets at least $\left\lceil \frac{n}{k} \right\rceil$ objects.

2) - - - - - - - - - gets at most $\left\lfloor \frac{n}{k} \right\rfloor$ objects.

**E.g.** Place 13 objects into 3 boxes. Then
$$\begin{cases} \text{1) Some box must get at least 5 objects.} \\ \text{2) - - - - - - - - - at most 4 objects.} \end{cases}$$

Pf: We'll prove (1) and leave the similar proof of (2) as an exercise.

(1): To prove that some box must contain at least $\lceil \frac{n}{k} \rceil$ objects.

We will suppose otherwise and derive a contradiction (i.e., we show that the opposite of the conclusion cannot possibly happen): "pf by contradiction".

Otherwise (if the desired conclusion doesn't hold), every box contains at most $\lceil \frac{n}{k} \rceil - 1$ objects.

So the $\underbrace{\text{total number of object in the boxes}}_{N}$ is at most $\left(\lceil \frac{n}{k} \rceil - 1\right) \cdot k$ object,

Since $\underline{\lceil \frac{n}{k} \rceil < \frac{n}{k} + 1}$, therefore it would follow that

$N < \left(\frac{n}{k}\right) \cdot k = n$, which is a contradiction, so we are done. □

## Examples:

(1) Gumball machine : many red, green, blue and white gumballs.

Each gumball costs 5 cents

Deal / Promotion : buy some gumball, and if 13 of them have the same color then you get \$5.

**Q:** What is the smallest number of gumballs you need to buy to ensure that you will make money on the deal ?

**Analysis:** The "worst case scenario is to buy gumballs to the point where you have 12 gumballs for each color $\longrightarrow 12 \times 4 = 48$ gumballs

$\downarrow$

49 should suffice!

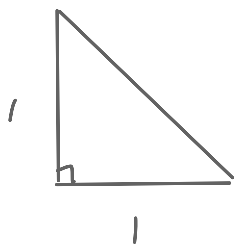
Soln: The smallest number of gumballs we should buy is 49, because

(1) it is not sufficient to buy 48 balls. Since we might get 12 balls of each of the four colors.

(2) On the other hand, if we buy 49 gumballs, then by the division principle some color will contribute at least $\left\lceil \frac{49}{4} \right\rceil = 13$ balls, so we will get the \$5 reward and make money. $\square$

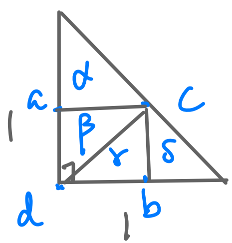
$(5 - 49 \times 0.05 = \$2.55)$

(2).




Suppose that nine points are placed at random in the triangle on the left. Show that at least three points of the nine form a triangle (possibly degenerate) whose area is $1/8$ or less.

Pf: Let a, b, c be the midpoints of the edges of the triangle as labeled, and let d be the vertex incidence to the right angle.




Connect ac, bc, dc as shown to partition the △ into four smaller △-regions, each with area $\frac{1\cdot 1}{2}\cdot\frac{1}{4}=\frac{1}{8}$. By the division principle, when we place 9 points in the big triangle, one of $\alpha, \beta, \gamma, \delta$ must get at least $\lceil \frac{9}{4} \rceil = 3$ of the 9 points. Those 3 points form a triangle of area less than $1/8$. □

## 3. More combinatorial identities (to finish Ch. 3)

(1). $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Pf: Consider taking $n$ objects from the two sets $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$. $\}$ 2n diff. objects in total.

One approach: Form the union $A \cup B$, which has $2n$ objects, and take $n$ objects

$\rightarrow \binom{2n}{n}$ way to do this.

Another approach: Take $k$ objects from $A$ and then $(n-k)$ objects from $B$, where $0 \le k \le n$.

$$\rightarrow \sum_{k=0}^{n} \binom{n}{k}\underline{\binom{n}{n-k}} = \sum_{k=0}^{n} \binom{n}{k}\underline{\binom{n}{k}} = \sum_{k=0}^{n} \binom{n}{k}^2 \text{ ways to do so.}$$

It follows that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

(2). $\binom{2n}{2} = 2\binom{n}{2} + n^2$ **Pf 1:** Show algebraically that LHS $= 2n^2 - n =$ RHS. $\square$

**Pf 2:** Consider picking 2 elts out of the union $A \cup B$ where $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ are sets with $n$ elts each.

There are clearly $\binom{|A \cup B|}{2} = \binom{2n}{2}$ to do so.

On the other hand, we could pick these 2 elts in one of the following ways:

(1) Pick them both from $A$. $\longrightarrow$ $\binom{n}{2}$ ways to do this

(2) . . — - . . $B$ $\rightarrow$ $\binom{n}{2}$ - - - - . . . .

(3) Pick one from $A$ and the other from $B$ (the only other option)

$\longrightarrow$ $\binom{n}{1} \cdot \binom{n}{1} = n^2$ ways to do this.

It follows that $\binom{2n}{2} = 2\binom{n}{2} + n^2$. $\square$