Math 2001. Lecture 10.

06. 14. 2022.

Last time:

- The inclusion-exclusion principle, $|A \cup B| = |A| + |B| - |A \cap B|$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

- Multiset combinations via the bars-and-stars method

Today:

- More bars-and-stars problems
- Multiset permutations (the word problem)
- Summary of counting problems and techniques
- The pigeonhole principle

1. Bars-and-stars exercises. $\to$ (often formulated without using multisets explicitly)

(a) # multisets of $\underline{\text{size } 4}$ made from $\underline{\{a, b, c, d, e, f\}}$ $= \binom{(6-1)+4}{4} = \binom{9}{4}$

6 regions. 5 bars

$* \; * \; * \; *$

e.g. $a \; b \; b \; e \longleftrightarrow * | ** ||| * |$

(b) # nonneg. int. solns. of $\underline{x+y+z=10}$ $= \binom{10+(3-1)}{3-1} = \binom{12}{2}$

10 stars, 3 regions, 2 bars

e.g. $***|**|***** \longleftrightarrow \underset{x}{3} + \underset{y}{2} + \underset{z}{5} = 10$

(c) # positive int. solns of $x+y+z=10$

$\Updownarrow \quad x' = x-1,\ y' = y-1,\ z' = z-1.$

$=$ # nonneg. int. solns of $x'+y'+z' = 7$ $= \binom{7+(3-1)}{3-1} = \binom{9}{2}$.

(d) We have 20 red balls, 20 blue balls, 20 green balls, 20 white balls. How many sets of 15 balls can we form out of these 80 balls?

Soln: Let x, y, z, w be the numbers of red, blue, green and white balls we use to form a set of 15 balls. Then it suffices to count the number of nonnegative int. solns of the eq. $x+y+z+w=15$.

This number should be $\binom{15+(4-1)}{4-1} = \binom{18}{3}$.

(e) 20 red, 20 blue, 20 green, 1 white, 1 black.

How many sets of 20 balls can we form?

Soln: We discuss cases and consider four (mutually exclusive) kinds of sets of 20 balls:

$(20r,\ 20b,\ 20g,\ 1w,\ 1b)$

$\quad x_1 \quad\ x_2 \quad\ x_3 \quad\ x_4 \quad\ x_5$

need $x_1+x_2+x_3+x_4+x_5=20.$

$\longleftrightarrow$ redundant $0\le x_1, x_2, x_3 \le 20$.

$0 \le x_4, x_5 \le 1.$ not redundant.

(i) The set contains both the white ball and the black ball.

→ We just need to get another 18 ball from the red, blue and green piles.

($x_4=x_5=1$. need $x_1+x_2+x_3=18$)

⇒ There are $\binom{18+(3-1)}{(3-1)} = \binom{20}{2}$.

(ii) The set contains the white ball but not the black ball. → $\binom{19+2}{2} = \binom{21}{2}$

(iii) ... black ... white ... $\binom{21}{2}$

(iv). The set contains neither the white ball nor the black ball. → $\binom{20+(3-1)}{3-1} = \binom{22}{2}$.

It follows that the desired number is $\binom{20}{2}+\binom{21}{2}\cdot 2+\binom{22}{2}$.

(f). What's the number of int. tuples (w, x, y, z) s.t.

$$0 \le w \le x \le y \le z \le 10 \ ?$$

$$\left(\begin{array}{lll} \text{e.g.} & (1, 2, 5, 6) \longrightarrow & \ast | \ast | \ast \ast \ast | \ast | \ast \ast \ast \ast \\ & (3, 6, 8, 8) \longleftarrow & \ast \ast \ast | \ast \ast \ast | \ast \ast | | \ast \ast \end{array} \right)$$

Soln: We note that such tuples are in bijection with bars-and-stars configurations containing 10 stars and 4 bars via the encoding

$$\text{a bars-and-stars diagram} \longrightarrow (x_1, x_2, x_3, x_4) \text{ where } x_i = \# \text{stars before the } i\text{th (leftmost) bar.}$$

It follows that the desired number of solns is $\binom{10+4}{4} = \binom{14}{4}$.

19). How about integer solns of $0 \le x < y < z \le 10$?

Consider the variables $y' = y - 1$ and $z' = z - \underset{\text{not } 1}{\underline{\underline{2}}}$ ⟵ motivation

so that the condition $x < y$ is equiv. to the condition that $x \le y'$

and the condition $y < z$ is equiv to the condition that $y' \le z'$

and the .... $z \le 10$ is .... $z' \le 8$.

It follows that # int. solns of $0 \le x < y < z \le 10$

$= \#$ int. solns of $0 \le x \le y' \le z' \le 8$.

$= \binom{8+3}{3}$ by the same reasoning as in the previous problem.

2. Multiset permutations

Q: Using (all) the letters of MISSISSIPPI,

how many different words/strings can we form?

Let's try a similar but easier problem: same question, but with 'BOOK'.

Key idea: distinguish the multiple occurrences of each letter first,

then compensate for the overcounting caused by the distinctions.

O_1, O_2.

$$\left\{\begin{array}{c}\text{permutations}\\ \text{of}\\ B, K, O_1, O_2\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c|c|c|c} BKO_1O_2 & BO_1KO_2 & & O_1O_2BK \\ BKO_2O_1 & BO_2KO_1 & \cdots & O_2O_1BK \\ \downarrow & \downarrow & & \downarrow \\ BKOO & BOKO & & OOBK \end{array}\right.$$

overcounting factor: $2 = \underbrace{2!}$

$\# \text{ perm. of the multiset } [B, K, O, O] = \dfrac{4!}{2!} = \dfrac{24}{2} = 12.$

$\# \text{ perm of } O_1, O_2.$

Another example: "BANANA"

If we distinguished the 2 Ns and the 3 As as N_1, N_2, A_1, A_2, A_3, then the letters $B, A_1, A_2, A_3, N_1, N_2$ have $6!$ permutations.

These permutations are in $3!\cdot 2!$ correspondence with the permutations of the multiset

$[B, A, A, A, N, N]$.

$$B A_1 A_2 A_3 N_1 N_2 \to BAAANN$$

$$B A_1 A_2 A_3 N_2 N_1 \to BAAANN$$

$$\vdots$$

$$B \underbrace{A_3 A_2 A_1}\, \underbrace{N_2 N_1} \to BAAANN$$

the perm of the As and of the Ns are "independent"

It follows that the number of spellings of BANANA is $\dfrac{6!}{3!\,2!}$

MISSISSIPPI ? $\dfrac{11!}{4!\,4!\,2!}$

We see that in general:

Prop: If A is a multiset with n elts (with mult.) and its elts have multiplicities $p_1, p_2, \dots, p_k$. Then the number of permutations of the elts of A is

$$\frac{n!}{p_1!\,p_2!\cdots p_k!}.$$

3. Summary of types of counting problems / techniques

Let X be a set $\{x_1, x_2, \cdots, x_n\}$ or a multiset $[x_1, x_2, \cdots, x_n]$ of n elts.

We have learned to count the following types of objects:

(1) Set permutation: X is a set, we take k elts out of X and line them up.

$$\#\text{ such lineups} = P(n,k) = n(n-1)\cdots(n-k+1).$$

In particular, if $k=n$, then $\#$ such lineups $= P(n,n) = n!$

$\downarrow$

rep. not allowed, order matters.

(2) Set combinations: X is a set, we take k elts of X to form a subset / comb.

$$\#\text{ comb.} = C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$\downarrow$

rep. not. allowed, order doesn't matter.

(3) Multiset permutations (the word problem):

X is a multiset and we line (all of) its elts up.

$$\# \text{ such perm.} = \frac{n!}{p_1! p_2! \cdots p_k!}$$ if ($|X| = n$) and the multiplicities of the elts of X are $p_1, p_2, \ldots, p_k$.

rep. is allowed, order matters.

(4) Multiset Combinations (bars-and-stars):

Y is a set of size n, we form a multiset of k elts using the elts of Y.

"alphabet"   "letters"

$$\# \text{ comb.} = \binom{k+(n-1)}{n-1} = \binom{k+(n-1)}{k}.$$

rep. is allowed, order doesn't matter.

Next time: the pigeonhole and division principles