Key Defs

(W a Coxeter group
$s_1, \dots s_n$ the simple refl.)

- $(i_1, \dots i_m)$ $1 \le i_n < n$ a word
- it is a reduced word for $w \in W$ if

  $$w = s_{i_1} \cdots s_{i_m}$$

  and the expression is as short as possible
- m is the length of w

Def

- A s.s.y.t is called strict if the rows are strictly increasing
- if $T = \boxed{i_1}\boxed{\;\cdots\;}\boxed{i_m}$, c

  $\mathrm{word}(T) = (i_1, \dots i_m)$
- if $T_1 \dots T_r$ are the rows of T, then $\mathrm{word}(T) = \mathrm{word}(T_r) \dots \mathrm{word}(T_1)$

The Algorithm

Assume S_K is an ascent for word(T)

- to insert K into $\boxed{i_1 \mid \cdots \mid i_s}$
- $K > i_s$
    - $\boxed{i_1 \mid \cdots \mid i_s \mid K}$
- $K \leq i_s$ ⟵ Note K an ascent $\Rightarrow K < i_s$
    - let u be the smallest or furthest left choice such that $K < i_u$
    - $K' = i_u$

The Algorithm Cont

- two cases
  - $u = 1$ or $u > 1$ and $i_{u-1} < K$
i_1	$\cdots$	i_{u-1}	K	i_{u+1}	$\cdots$	i_s
K'						
  - $u > 1$ and $i_{u-1} = K$
i_1	$\cdots$	i_{u-1}	i_u	i_{u+1}	$\cdots$	i_s
K'						

Notice the row doesn't change

- $i_{u-1} = K$
- $i_u = K' = K+1$

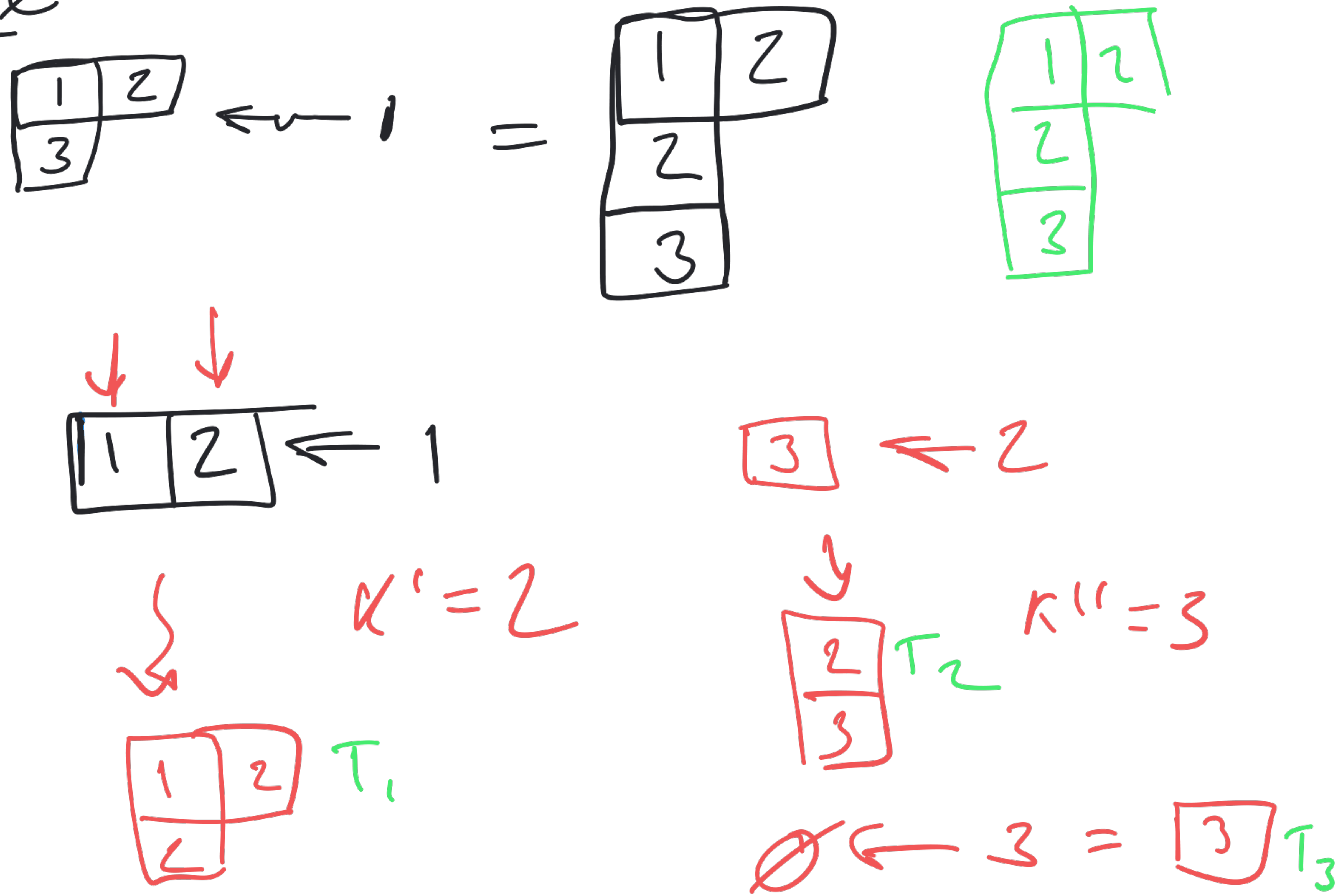
The Algorithm Cont

- the algorithm continues by inserting K' into the next row

Ex

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \leftarrow 1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

Ex



$$\boxed{2 \mid 4} \leftarrow 1$$

Lemma 7.16

If T is a reduced word and K is an ascent, then

$$w(T \leftarrow K) = w(T)\, s_K$$

pf (sketch)

Reduces to T a row

3 cases

- K is appended
- (7.4)
- (7.5)

B_K' word(T)

$= \text{word}(T)\, s_K$

Edelman - Greene

- $\mathbf{i} = (i_1 \ldots i_m)$ a reduced word for w
- $P(\mathbf{i}) = \emptyset \leftarrow i_1 \leftarrow \ldots \leftarrow i_m$
- $Q(\mathbf{i})$ the recording tableau

Ex

- $i = (2, 3, 2, 1, 2)$
- $$P(i) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$
- $$Q(i) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

Thm

- w a permutation.
- the map

$$i \longmapsto (P(i), Q(i))$$

is a bijection between reduced words for w and pairs of tableaux of the same shape such that P is reduced and s.s., Q is standard and $w = w(P)$.

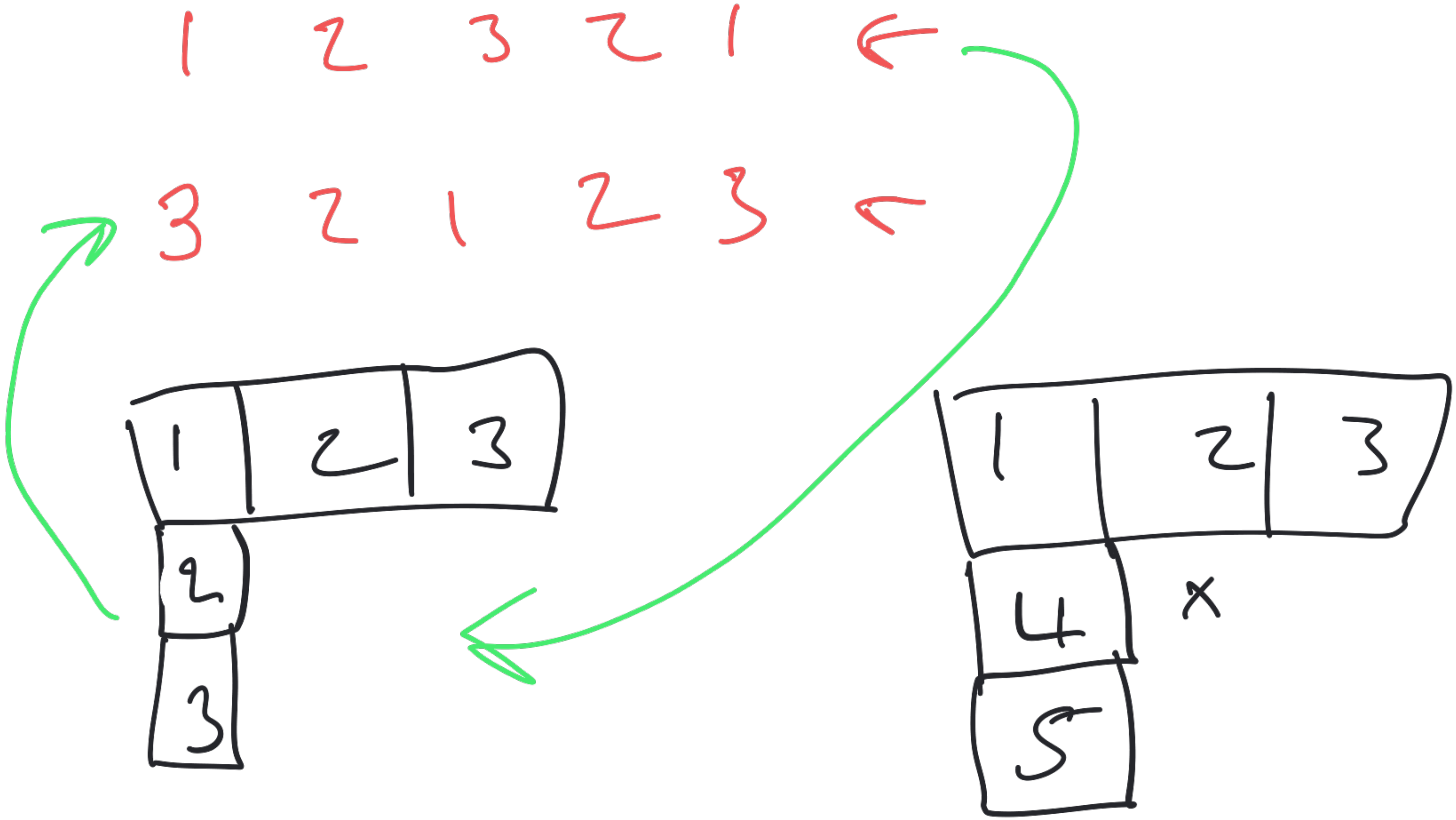
Exercise 7.3. For the six permutations in S_3, compute the insertion tableau P and the recording tableau Q. Which permutations have the same P-tableau (resp. Q-tableau)?

$\varnothing$ $\quad s_1 \quad s_1 s_2 \quad s_1 s_2 s_1$

$\quad s_2 \quad s_2 s_1 \quad s_2 s_1 s_2$

$\varnothing \ \varnothing \quad \boxed{1} \ \boxed{1} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

$\boxed{2} \ \boxed{1} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$


1 2 3 2 1
3 2 1 2 3
1 2 3
2
3
1 2 3
4
5
x

Chapter 8: The Platic Monoid

April 3, 2020

Notation: $\mathbb{B}$: standard $GL(n)$ crystal
$\mathcal{B}_\lambda$: crystal of tableaux (embedded with Row Reading)

Goal: Understand the connected components of $\mathbb{B}^{\otimes k}$

Things to "recall":

- Normal crystals: "type free" idea of a Stembridge crystal
- Isomorphisms of connected normal crystals are unique.
- B_λ are Stembridge - get canonical reps of isomorphism classes with highest weight λ!
- All components of $\mathbb{B}^{\otimes k}$ are isomorphic to B_λ for some λ.

Plan: For a connected component C of $\mathbb{B}^{\otimes k}$, find λ so that $C \cong B_\lambda$ and give this isomorphism reading word: $x_1 \cdots x_k$

Main Result: Let $x = x_1 \otimes x_2 \otimes \cdots \otimes x_k \in \mathcal{B}^{\otimes k}$.

Define $P(x)$ (SSYT) and $Q(x)$ (SYT) by

$$(P(x), Q(x)) \overset{RSK}{=} \emptyset \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_k$$

Then:

- $\lambda = sh(P(x))$ determines isomorphism type $\mathcal{B}_\lambda$ for the conn. comp of x.

Plactic class $\longrightarrow$

- $P(x)$ determines the image of x under the isomorphism (reading word)

Coplactic class $\longrightarrow$

- $Q(x)$ determines the connected component

Describing the Isomorphism

Define plactic equivalence on elements of a connected normal crystal by

$$x \equiv y \quad \text{if} \quad \begin{array}{ccc} C_1 & \xrightarrow{\sim} & C_2 \\ x & \longmapsto & y \end{array}$$

Define Knuth equivalence on words by elementary operations on consecutive sequences of length 3 by

$$bac \equiv_K bca \qquad \text{if} \qquad a < b \le c$$

$$acb \equiv_K cab \qquad \text{if} \qquad a \le b < c$$

---

1 2 3   3 1 2

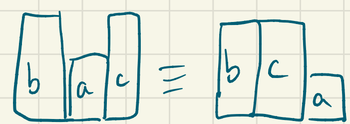
$1\,3\,2 \xleftrightarrow{\equiv_K} 3\,1\,2$

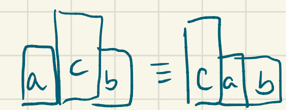
1 3 2   3 2 1

• 2 1 3

$\equiv_K$

2 3 1





Important Results

Prop: In type A, the highest weight vectors in $B^{\otimes k}$ are Yamanouchi words:

↳ in all final segments, the # of i's $\geq$ the # of $(i+1)$'s

Thm: In type A, Knuth equivalence implies plactic equivalence.

Things we can try:

- Compute the Knuth map for $\mathcal{B}_{(2,1)} \cong \mathcal{B}'_{(2,1)}$ from figure 3.3

  — show the same for other $\mathcal{B}^{\otimes k}$

- Exercise 8.2 -postpone

Knuth map for $\mathcal{B}_{(2,1)} \cong \mathcal{B}'_{(2,1)}$:

2 1 1

2 1 2   3 1 1

2 1 3   3 1 2

3 1 3   3 2 2

3 2 3

a c b

1 2 1

2 2 1   1 3 1

2 3 1   1 3 2

3 3 1   2 3 2

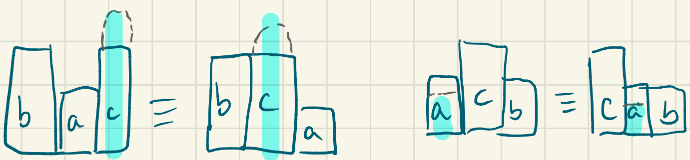
3 3 2

c a b

2 1 1

1 2 1 $\not\equiv$ 1 1 2

b c a



"If you have a tie, the right one should be read as 'bigger'"

Question: how is this related to 321 ⟷ 132 avoiding permutations

Next time:

- 4-fold tensor product crystal
- Other examples → • column reading + row reading should be Knuth equivalent
  are
- If time allows, 8.3, 8.4

- Other apps of Knuth equiv?