Chapter 7: Insertion Algorithms

Chapter is mostly background. We will discuss the relevance to Type A crystals based on notes from Daniel Bump's website.

§7.1 RSK Algorithm:

Example: Group algebra of symmetric group.

$$\mathbb{C}[S_k] \cong \bigoplus_{\lambda \vdash k} \pi_\lambda^{S_k} \otimes \pi_\lambda^{S_k}$$

Schur-Weyl Duality: $GL_n(\mathbb{C})$ acts on $V = \mathbb{C}^n$ by standard representation.

Commuting actions of S_k and $GL_n(\mathbb{C})$ on $V^{\otimes k}$,

this module decomposes:
$$V^{\otimes k} = \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} \pi_\lambda^{S_k} \otimes \pi_\lambda^{GL_n(\mathbb{C})}$$

Cauchy Identity: $GL_r(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $\mathbb{C}^r \otimes \mathbb{C}^n$ by tensor of the standard reps.

Equivalent to action on $Mat_{r \times n}(\mathbb{C})$
$$(g,h): X \mapsto g \cdot X \cdot {}^T h$$

The symmetric algebra decomposes
$$\bigvee(Mat_{r\times n}(\mathbb{C})) \cong \bigoplus_{\ell(\lambda) \leq \min(r,n)} \pi_\lambda^{GL_r} \otimes \pi_\lambda^{GL_n}$$

Letting α_i, β_i be eigenvalues of g, h resp., $tr(g,h) = \prod_{i,j} (1-\alpha_i\beta_j)^{-1} = \sum_\lambda s_\lambda(\alpha) s_\lambda(\beta)$, where s_λ is the Schur function.

Dual Cauchy Identity: $$\bigwedge(Mat_{r\times n}(\mathbb{C})) \cong \bigoplus_{\substack{\ell(\lambda) \leq r \\ \ell(\lambda') \leq n}} \pi_\lambda^{GL_r} \otimes \pi_{\lambda'}^{GL_n}$$

Summed over partitions λ fitting into an $r \times n$ box, λ' the conjugate partition.

Taking traces gives $\prod_{i,j}(1+\alpha_i\beta_j) = \sum_\lambda s_\lambda(\alpha) s_{\lambda'}(\beta)$.

Combinatorial analogues based on Schensted insertion. First, some notation:

- For tableaux T, $T(i,j)$ is the box in row i, column j.
- Given μ, λ partitions s.t. $YD(\mu) \supseteq YD(\lambda)$, say that (μ, λ) is a <u>skew shape</u>, denoted μ/λ.

  Then $YD(\mu/\lambda) = YD(\mu) \setminus YD(\lambda)$ as a set-theoretic difference.

Schensted Insertion:

Given a SST T of shape λ, and i, obtain $T \leftarrow i$ by...

① If i is at least the last entry of the first row, put i at the end of that row.

② Else, we bump the smallest j in Row 1 s.t. $j > i$:

Replace the leftmost j with i, then try inserting j into row 2.

③ Repeat until we are able to place our final bumped entry.

Note i will always be in the first row of $T \leftarrow i$.

Example: $T = \begin{array}{ccccccc} 1 & 1 & 1 & 2 & 2 & 4 & 4 \\ 2 & 2 & 3 & 3 & 4 & & \\ 3 & 3 & & & & & \\ 4 & & & & & & \end{array}$ $\xrightarrow{\leftarrow 1}$

1 bumps 2:

$\begin{array}{ccccccc} 1 & 1 & 1 & 2 & 2 & 4 & 4 \\ 2 & 2 & 3 & 3 & 4 & & \\ 3 & 3 & & & & & \\ 4 & & & & & & \end{array}$ $\rightarrow 1\ 1\ 1\ 1\ 2\ 4\ 4$, $\rightarrow 2\ 2\ 2\ 3\ 4$

2 bumps 3

3 fits into row 3:

$T \leftarrow 1 = \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 4 & 4 \\ 2 & 2 & 2 & 3 & 4 & & \\ 3 & 3 & 3 & & & & \\ 4 & & & & & & \end{array}$

"landing location"

Shaded: "bumping path"

"new location"

Fact: If T is semistandard, so is $T \leftarrow i$.

Proof: Obvious by design.

RSK algorithm is a collection of bijections:

(7.7) $\alpha \in S_k$ $\longleftrightarrow$ pairs (P,Q) of SYT with shape $\lambda \vdash k$ in alphabet $[k]$

more structure

(7.8) words $w = (i_1, \dots, i_k)$ in alphabet $[n]$ $\longleftrightarrow$ pairs (P,Q) of SSYT of shape $\lambda \vdash k$, where P is semistandard w/ entries in $[n]$, and Q is standard in $[k]$.

less structure

(7.9) $r \times n$ matrices with entries in $\{0, 1, \dots, n\}$ $\longleftrightarrow$ pairs (P,Q) of tableaux of shape $\lambda \vdash k$, P has entries in $[r]$, Q has entries in $[n]$.

(7.10) $r \times n$ matrices w/ entries in $\{0,1\}$ $\longleftrightarrow$ pairs (P,Q) of tableaux s.t. P, Q have conjugate shape, P has alphabet $[r]$, Q has alphabet $[n]$.

7.8 Given word $w = (i_1, \dots, i_k)$, get insertion tableau P:

$P = \varnothing \leftarrow i_1 \leftarrow i_2 \leftarrow \cdots \leftarrow i_k$ and recording tableau Q, whose entry r is the new location of the last insertion $\varnothing \leftarrow i_1 \leftarrow \cdots \leftarrow i_r$. (Recovers the ordering of w).

Example: $w = (3,1,2,1,4)$.

P: $\begin{array}{c} 3 \end{array} \rightarrow \begin{array}{c} 1 \\ 3 \end{array} \rightarrow \begin{array}{cc} 1 & 2 \\ 3 & \end{array} \rightarrow \begin{array}{cc} 1 & 1 \\ 2 & \\ 3 & \end{array} \rightarrow \begin{array}{ccc} 1 & 1 & 4 \\ 2 & & \\ 3 & & \end{array}$

$w \mapsto (P,Q)$ is the RSK map.

Q: Shape of P, records order boxes were added: $\begin{array}{ccc} 1 & 3 & 5 \\ 2 & & \\ 4 & & \end{array}$

7.8 Corresponds to Schur-Weyl duality:

$$|\{\text{words of length } k \text{ in } [n]\}| = n^k = \dim(V^{\otimes k})$$

If $\lambda \vdash k$ has $\ell(\lambda) \le n$, then the # of SSYT P of shape λ in $[n]$ is the degree of $\pi_\lambda^{GL_n(\mathbb{C})}$, and # of SYT Q in shape λ is degree of $\pi_\lambda^{S_k}$ in the Schur-Weyl duality isomorphism.

- To see the bijection in 7.7, set $n=k$ and observe a word w is a permutation of $[k]$ if and only if P is standard.
(If P were semistandard then w would have a repeated entry).

Bijection from 7.9: Given $X \in Mat_{r\times n}(\mathbb{N})$, make a two-rowed array A whose columns $\binom{i}{j}$ have $i\in[r]$, $j\in[n]$ and s.t. A has x_{ij} columns of the form $\binom{i}{j}$.

Example: $X = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$,

Obtain tableaux (P,Q) by:
P = inserting the bottom row (left-to-right) to $\emptyset$
Q = put an i in the new location formed by inserting j.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 1 & 1 & 3 & 3 \end{pmatrix}$$

$$P = \underbrace{\emptyset \leftarrow 1 \leftarrow 2 \leftarrow 2 \leftarrow 2}_{\text{all have } i=1 \text{ in corresponding } \binom{i}{j}} \leftarrow 1 \leftarrow 1 \leftarrow 3 \leftarrow 3 = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$$

these all have $i=2$ in corresponding $\binom{i}{j}$

Intermediate step: $\begin{cases} P' = \emptyset \leftarrow 1 \leftarrow 2 \leftarrow 2 \leftarrow 2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline\end{array} \\ Q' = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline\end{array} \end{cases}$

$$Q = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$$

Note Q is semistandard

- **How to reconstruct X from (P,Q):** Suffices to reconstruct A and count columns. (Let $A \leftrightsquigarrow (P,Q)$, $A' \leftrightsquigarrow (P',Q')$)

Let $\binom{i}{j}$ be the rightmost column, and get A' by deleting $\binom{i}{j}$ from A.

Step 1 | Determine the last new location in insertion tableau P (given by $P' \leftarrow j$).

As Q is semistandard, this is given by the rightmost location of the largest entry in Q.

Step 2 | The new location is at the end of a row k, so the bumping path is
$(1,c_1) \to (2,c_2) \to \dots \to (k,c_k)$

where the c_i are decreasing (since columns are strictly increasing).

- c_{k-1} is the largest integer s.t. $P(k-1,c_{k-1}) < P(k,c_k)$.

- Entries of P' not on the bumping path are same as P.

Entries on bumping path are $P'(i,c_i) = P(i+1,c_{i+1})$

Example:

$$P = \begin{array}{cccccc} 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 & & \\ 3 & 3 & & & & \\ 4 & & & & & \end{array} \quad \leftarrow \text{new location } (4,1).$$

$$P(3,2) < P(4,1)$$

$$P(2,2) < P(3,2)$$

$$\underbrace{P(1,3)} < P(2,2)$$

$$j = P(1,3) = 1.$$

Theorem 7.14 is a (long!) proof of symmetry in RSK: if $X \in \mathrm{Mat}_{r \times n}(\mathbb{N})$ is s.t. $X \rightsquigarrow (P, Q)$ then ${}^tX \rightsquigarrow (Q, P)$.

§7.2 Dual RSK

- A tableaux is <u>dual semi standard</u> if the rows are strictly incre and if columns are weakly increasing. It is <u>dual standard</u> if columns strictly increase.

<u>Dual Schensted insertion</u> $T \Leftarrow j$ is defined so that $T \Leftarrow j$ remains row-strict.

- If $j \leq$ all entries in row 1, it <u>bumps</u> the largest such entry.

Dual RSK gives a bijection between $r \times n$ matrices with entries $\{0,1\}$ and pairs of tableaux (P, Q) that are dual semistandard in $[r]$, $[n]$ respectively, using the same method of constructing A with colums $\binom{i}{j}$ appearing x_{ij} times.

- Dual RSK doesn't seem immediately relevant.

§7.3 Edelman-Greene Insertion

RSK variant used in studying reduced words for elements of S_n.

- Fix n, let W be a Coxeter group w/simple reflections $s_1, \dots, s_{n-1}$.

A <u>word</u> $(i_1, \dots i_r)$ is a <u>reduced word</u> for $w \in W$ if $w = s_{i_1} \cdots s_{i_r}$ is as short as possible.

- Given a tableau T with rows $T_1, \dots, T_r$, obtain a word

$$\mathrm{word}(T) \text{ by row-reading } T \text{ from the bottom up.}$$

We say T is <u>reduced</u> if $w = \mathrm{word}(T)$ is a reduced word, and write $w(T) = w$.

- <u>EG Insertion</u>: Assume T is reduced, fix an integer k such that

k is a right ascent for $w = \mathrm{word}(T)$, so that $\mathrm{length}(w s_k) = \mathrm{length}(w) + 1$.

Obtain $T \leftsquigarrow k$ by:

① If $T = \boxed{i_1 \ i_2 \cdots i_s}$ and $k > i_s$, $T \leftsquigarrow k = \boxed{i_1 \ i_2 \cdots i_s \ k}$

If $k \leq i_s$, then since k is an ascent for w, $k < i_s$. Fix minimal u s.t. $k < i_u$

and define $k' = i_u$.

<u>Cases</u>| ⓐ $u = 1$ or $i_{u-1} < k$. Then $T \leftsquigarrow k = T \leftarrow k$ as in standard RSK.

ⓑ $u > 1$ and $i_{u-1} = k$. Then $i_u = k+1$, and

$$T \leftsquigarrow k = \begin{array}{|c|} \hline i_1 \cdots i_{u-1} \ i_u \cdots i_s \\ \hline k' \\ \hline \end{array}$$ Note the top row is unchanged.

In both of these, k' was "<u>bumped</u>".

② If T is <u>not</u> a row, let $T_1, \dots, T_r$ be its rows. Some of the rows will be replaced.

- If $T_1 \leftsquigarrow k$ is a row, set $T_1' = T_1 \leftsquigarrow k$ and finish: $T = T_1', T_2, \dots, T_r$.

- If not, bump an entry k_1, and proceed inductively, trying to place the bumped entry.

replace T_1 by a row T_1' s.t. $(T_1 \leftsquigarrow k) = (k_1 \leftsquigarrow T_1')$

<u>EG "RSK"</u>| Given reduced word $\mathbf{i} = (i_1, \dots, i_m)$ for $w \in S_n$, define

$$P(\mathbf{i}) = \emptyset \leftsquigarrow i_1 \leftsquigarrow i_2 \leftsquigarrow \cdots \leftsquigarrow i_m$$

and $Q(\mathbf{i})$ the recording tableaux as in standard RSK.

Then given fixed $w \in S_n$, there is a bijection $\mathbf{i} \longleftrightarrow (P(\mathbf{i}), Q(\mathbf{i}))$ between

reduced words for w and pairs (P, Q) s.t. P is reduced + semistandard s.t. $w = w(P)$ and Q standard.

<u>Example</u>: $\mathbf{i} = (2,3,2,1,2)$ $P(\mathbf{i}) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$ $Q(\mathbf{i}) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}$

- How does any of this relate to crystals? (c.f. Daniel Bump's website).

Given a crystal $\mathcal{B}_r$ of a row tableaux, embed $\mathcal{B}_r \hookrightarrow \mathbb{B}^{\otimes r}$

$$\boxed{i_1\ i_2\ \cdots\ i_r} \longrightarrow i_r \otimes \cdots \otimes i_2 \otimes i_1$$

Given a crystal $\mathcal{B}_\lambda$ of a partition λ with k parts, send the tableaux $T = T_1, T_2, \dots T_k \longleftrightarrow T_1 \otimes T_2 \otimes \cdots \otimes T_k$.

<u>Proposition:</u> There is an isomorphism $\mathcal{B}_{(k)} \otimes \mathbb{B} \cong \mathcal{B}_{(k+1)} \oplus \mathcal{B}_{(k,1)}$, wherein if $T \in \mathcal{B}_{(k)}$ is a row, then $T \otimes \boxed{j}$ corresponds to $T \leftarrow j$.