

Chapter 7: Insertion Algorithms

Chapter is mostly background. We will discuss the relevance to Type A crystals based on notes from Daniel Bump's website.

§ 7.1 RSK Algorithm:

Example: Group algebra of symmetric group.

$$\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} \pi_{\lambda}^{S_n} \otimes \pi_{\lambda}^{S_n}$$

Schur-Weyl Duality: $GL_n(\mathbb{C})$ acts on $V = \mathbb{C}^n$ by standard representation.

Commuting actions of S_k and $GL_n(\mathbb{C})$ on $V^{\otimes k}$,

this module decomposes: $V^{\otimes k} = \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} \pi_{\lambda}^{S_k} \otimes \pi_{\lambda}^{GL_n(\mathbb{C})}$

Cauchy Identity: $GL_r(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $\mathbb{C}^r \otimes \mathbb{C}^n$ by tensor of the standard repr.

Equivalent to action on $\text{Mat}_{r \times n}(\mathbb{C})$

$$(g, h): X \mapsto g \cdot X \cdot h^T$$

The symmetric algebra decomposes

$$V(\text{Mat}_{r \times n}(\mathbb{C})) \cong \bigoplus_{\lambda \vdash \min(r, n)} \pi_{\lambda}^{GL_r} \otimes \pi_{\lambda}^{GL_n}$$

Letting α_i, β_j be eigenvalues of g, h resp, $\text{tr}(g, h) = \prod_{i,j} (1 - \alpha_i \beta_j)^{-1} = \sum_{\lambda} s_{\lambda}(\alpha) s_{\lambda}(\beta)$,
where s_{λ} is the Schur function.

Dual Cauchy Identity: $\Lambda(\text{Mat}_{r \times n}(\mathbb{C})) \cong \bigoplus_{\substack{\lambda'(\lambda') \leq r \\ \ell(\lambda') \leq n}} \pi_{\lambda'}^{GL_r} \otimes \pi_{\lambda'}^{GL_n}$

Summed over partitions λ fitting into an $r \times n$ box, λ' the conjugate partition.

$$\text{Taking traces gives } \prod_{i,j} (1 + \alpha_i \beta_j) = \sum_{\lambda} s_{\lambda}(\alpha) s_{\lambda'}(\beta).$$

Combinatorial analogues based on Schensted insertion. First, some notation:

• For tableaux T , $T(i, j)$ is the box in row i , column j .

• Given μ, λ partitions s.t. $YD(\mu) \supseteq YD(\lambda)$, say that (μ/λ) is a skew shape, denoted μ/λ .

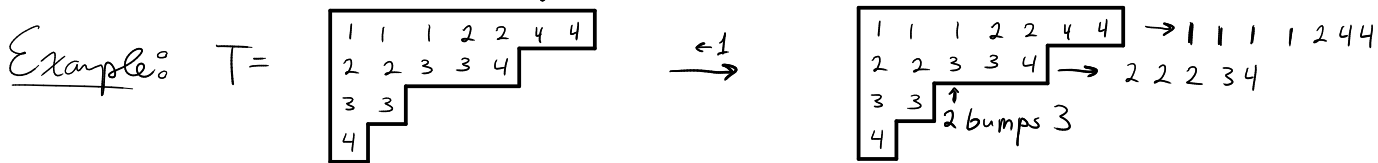
Then $YD(\mu/\lambda) = YD(\mu) \setminus YD(\lambda)$ as a set-theoretic difference.

Schensted Insertions

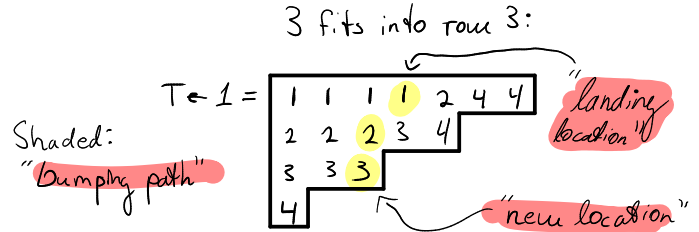
Given a SST T of shape λ , and i , obtain $T \leftarrow i$ by...

- ① If i is at least the last entry of the first row, put i at the end of that row.
- ② Else, we bump the smallest j in Row 1 s.t. $j > i$:
Replace the leftmost j with i , then try inserting j into row 2.
- ③ Repeat until we are able to place our final bumped entry.

Note i will always be in the first row of $T \leftarrow i$.



Fact: If T is semistandard, so is $T \leftarrow i$.
Proof: Obvious by design.



RSK algorithm is a collection of bijections:

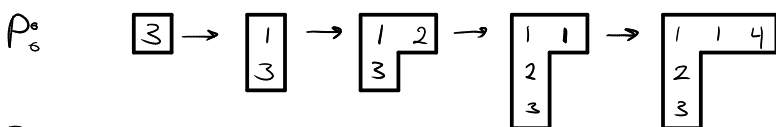
- ⑦.7 $\alpha \in S_k \iff$ pairs (P, Q) of SYT with shape $\lambda \vdash k$ in alphabet $[k]$ more structure
- ⑦.8 words $w = (i_1, \dots, i_k)$ in alphabet $[n] \iff$ pairs (P, Q) of SSYT of shape $\lambda \vdash k$, where P is semistandard w/ entries in $[n]$, and Q is standard in $[k]$.
- ⑦.9 $r \times n$ matrices with entries in $\{0, 1, \dots, n\} \iff$ pairs (P, Q) of tableaux of shape $\lambda \vdash k$, P has entries in $[r]$, Q has entries in $[n]$.
- ⑦.10 $r \times n$ matrices w/ entries in $\{0, 1\} \iff$ pairs (P, Q) of tableaux s.t. P, Q have conjugate shape, P has alphabet $[r]$, Q has alphabet $[n]$. less structure

⑦.8 | Given word $w = (i_1, \dots, i_k)$, get insertion tableau P :

$P = \emptyset \leftarrow i_1 \leftarrow i_2 \leftarrow \dots \leftarrow i_k$ and recording tableau Q , whose entry r is

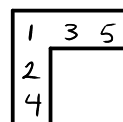
the new location of the last insertion $\emptyset \leftarrow i_1 \leftarrow \dots \leftarrow i_r$.
(Recovers the ordering of w).

Example: $w = (3, 1, 2, 1, 4)$.



$w \mapsto (P, Q)$ is the RSK map.

$Q:$ Shape of P , records order boxes were added:



7.8 Corresponds to Schur-Weyl duality:

$$|\{\text{words of length } k \text{ in } [n]\}| = n^k = \dim(V^{\otimes k})$$

If $\lambda \vdash k$ has $l(\lambda) \leq n$, then the # of SSYT P of shape λ in $[n]$ is the degree of $\pi_\lambda^{GL_n(\mathbb{C})}$, and # of SYT Q in shape λ is degree of $\pi_\lambda^{S_n}$ in the Schur-Weyl duality isomorphism.

• To see the bijection in (7.7), set $n=k$ and observe a word w is a permutation of $[k]$ if and only if P is standard.
(If P were semistandard then w would have a repeated entry).

Bijection from (7.9): Given $X \in \text{Mat}_{r \times n}(\mathbb{N})$, make a two-rowed array A

whose columns $\binom{i}{j}$ have $i \in [r], j \in [n]$ and s.t. A has x_{ij} columns of the form $\binom{i}{j}$.

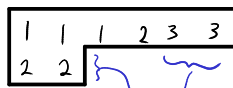
Example: $X = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 1 & 1 & 3 & 3 \end{pmatrix}$$

$$P = \emptyset \leftarrow 1 \leftarrow 2 \leftarrow 2 \leftarrow 2 \leftarrow 1 \leftarrow 1 \leftarrow 3 \leftarrow 3 =$$

all have $i=1$ in corresponding $\binom{i}{j}$

Obtain tableaux (P, Q) by:
 $P =$ inserting the bottom row (left-to-right) to \emptyset
 $Q =$ put an i in the new location formed by inserting j .



these all have $i=2$ in corresponding $\binom{i}{j}$

Intermediate step: $\begin{cases} P = \emptyset \leftarrow 1 \leftarrow 2 \leftarrow 2 \leftarrow 2 = \boxed{1 \ 2 \ 2 \ 2} \\ Q = \boxed{1 \ 1 \ 1 \ 1} \end{cases}$

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & & & & \end{pmatrix}$$

Note Q is semistandard.

• How to reconstruct X from (P, Q) : Suffices to reconstruct A and count columns. (Let $A \leftrightarrow (P, Q)$, $A' \leftrightarrow (P, Q')$)

Let $\binom{i}{j}$ be the rightmost column, and get A' by deleting $\binom{i}{j}$ from A .

Step 1 Determine the last new location in insertion tableau P (given by $P \leftarrow j$).

As Q is semistandard, this is given by the rightmost location of the largest entry in Q .

Step 2 The new location is at the end of a row k , so the bumping path is $(1, c_1) \rightarrow (2, c_2) \rightarrow \dots \rightarrow (k, c_k)$

where the c_i are decreasing (since columns are strictly increasing).

• c_{k-1} is the largest integer s.t. $P(k-1, c_{k-1}) < P(k, c_k)$.

• Entries of P not on the bumping path are same as P .

Entries on bumping path are $P(i, c_i) - P(i+1, c_{i+1})$

Example:

$P =$

1	1	1	2	3	4
2	2	4	4		
3	3				
4					

← new location. (4,1).

$$P(3,2) < P(4,1)$$

$$P(2,2) < P(3,2)$$

$$P(1,3) < P(2,2)$$

$$j = P(1,3) = 1.$$

Theorem 7.14 is a ^(long!) proof of symmetry in RSK: if $\chi \in \text{Mat}_{\text{rn}}(\mathbb{N})$ is s.t.
 $\chi \rightsquigarrow (P, Q)$ then ${}^t\chi \rightsquigarrow (Q, P)$.

§7.2 Dual RSK

- A tableau is dual semi standard if the rows are strictly increasing and if columns are weakly increasing. It is dual standard if columns strictly increase.

Dual Schensted insertion $T \leftarrow j$ is defined so that $T \leftarrow j$ remains row-strict.

- If $j \leq$ all entries in row 1, it bumps the largest such entry.

Dual RSK gives a bijection between $r \times n$ matrices with entries $\{0,1\}$ and pairs of tableaux (P, Q) that are dual semi standard in $[r], [n]$ respectively, using the same method of constructing A with columns $\begin{pmatrix} i \\ j \end{pmatrix}$ appearing x_{ij} times.

- Dual RSK doesn't seem immediately relevant.

§7.3 Edelman-Greene Insertion

RSK variant used in studying reduced words for elements of S_n .

- Fix n , let W be a Coxeter group w/simple reflections s_1, \dots, s_{n-1} .

A word (i_1, \dots, i_r) is a reduced word for $w \in W$ if $w = s_{i_1} \dots s_{i_r}$ is as short as possible.

• Given a tableau T with rows T_1, \dots, T_r , obtain a word

$\text{word}(T)$ by row-reading T from the bottom up.

We say T is reduced if $w = \text{word}(T)$ is a reduced word, and write $w(T) = w$.

• EG Insertion: Assume T is reduced, fix an integer k such that

k is a right ascent for $w = \text{word}(T)$, so that $\text{length}(w s_k) = \text{length}(w) + 1$.

Obtain $T \leftarrow k$ by:

① If $T = \boxed{i_1 \ i_2 \ \dots \ i_s}$ and $k > i_s$, $T \leftarrow k = \boxed{i_1 \ i_2 \ \dots \ i_s \ k}$

If $k \leq i_s$, then since k is an ascent for w , $k < i_s$. Fix minimal u s.t. $k < i_u$ and define $k' = i_u$.

Cases | ② $u=1$ or $i_{u-1} < k$. Then $T \leftarrow k = T \leftarrow k$ as in standard RSK.

③ $u > 1$ and $i_{u-1} = k$. Then $i_u = k+1$, and

$T \leftarrow k = \boxed{i_1 \ \dots \ i_{u-1} \ i_u \ \dots \ i_s}$ Note the top row is unchanged.
 $\quad \quad \quad \boxed{k'}$

In both of these, k' was "bumped".

② If T is not a row, let T_1, \dots, T_r be its rows. Some of the rows will be replaced.

• If $T_1 \leftarrow k$ is a row, set $T'_1 = T_1 \leftarrow k$ and finish: $T = T'_1, T_2, \dots, T_r$.

- If not, bump an entry k_{i_1} and proceed inductively, trying to place the bumped entry.

replace T_1 by a row T'_1 s.t. $(T_1 \leftarrow k) = (k' \leftarrow T'_1)$

EG "RSK" | Given reduced word $\mathbf{i} = (i_1, \dots, i_m)$ for $w \in S_n$, define

$$P(\mathbf{i}) = \emptyset \leftarrow i_1 \leftarrow i_2 \leftarrow \dots \leftarrow i_m$$

and $Q(\mathbf{i})$ the recording tableaux as in standard RSK.

Then given fixed $w \in S_n$, there is a bijection $\mathbf{i} \leftrightarrow (P(\mathbf{i}), Q(\mathbf{i}))$ between

reduced words for w and pairs (P, Q) s.t. P is reduced + semistandard s.t. $w = w(P)$ and Q standard.

Example: $\mathbf{i} = (2, 3, 2, 1, 2)$

$$P(\mathbf{i}) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$$

$$Q(\mathbf{i}) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline \end{array}$$

• How does any of this relate to crystals? (c.f. Daniel Bump's website).

Given a crystal \mathcal{B}_r of a row tableaux, embed $\mathcal{B}_r \hookrightarrow \mathcal{B}^{\otimes r}$

$$\boxed{i_1, i_2, \dots, i_r} \rightarrow i_r \otimes \dots \otimes i_2 \otimes i_1$$

Given a crystal \mathcal{B}_λ of a partition λ with k parts, send

$$\text{the tableaux } T = T_1, T_2, \dots, T_k \longleftrightarrow T_1 \otimes T_2 \otimes \dots \otimes T_k.$$

Proposition There is an isomorphism $\mathcal{B}_{(k)} \otimes \mathcal{B} \cong \mathcal{B}_{(k+1)} \oplus \mathcal{B}_{(k,1)}$,

wherein if $T \in \mathcal{B}_{(k)}$ is a row, then $T \otimes \square$ corresponds to $T \leftarrow j$.