Chapter 6. Crystals of Tableaux II.

Review: (1) $\mathbb{B}$. $\quad 1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{r} r+1$

$\uparrow$ the $GL(r+1)$ crystal (A_r)

$\mathrm{wt}(i) = e_i$, seminormal $\longrightarrow$ ε_i, φ_i can be read from root strings, $\varphi_i(j) = \delta_{ij}$

$\varepsilon_i(j) = \delta_{i,j-1}$

(2) The tensor rule

2-fold : $$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{if } \varphi_i(y) \leq \varepsilon_i(x) \\ x \otimes f_i(y) & \text{if } \varphi_i(y) > \varepsilon_i(x) \end{cases}$$

n-fold : the formula generalizes the 2-fold formula and can be understood by a signature rule : things become particularly easy for $\mathbb{B}^{\otimes k}$. (Sec. 2.4.)

13) Crystal of row tableaux: $B_{(k)}$

— $B_{(k)}$ will be $GL(n)$-crystals w/ $n=r+1$ in the previous notation.

— elts of $B_{(k)}$ are length/width-k row tableaux filled w/ entries in $[n]$.

$B^{(n)}_{(k)}$ → $n=r+1$ is the alphabet size, often omitted in the notation
→ k is the tableaux size

actually defined directly in tableaux language

— weight record numbers of occurences of entries.

— f_i: change rightmost i to $i+1$ if it exists; send to 0 otherwise.

e.g. $\mathbb{B} = \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$ $GL(3)$ crystal $n=3$. $r=2$.

$$\mathbb{B}\otimes\mathbb{B} = \left(\begin{tikzcd} & & \boxed{2}\otimes\boxed{2} \arrow[dr, "2"] & & \\ \boxed{1}\otimes\boxed{1} \arrow[r, "1"] & \boxed{1}\otimes\boxed{2} \arrow[ur, "1"] \arrow[dr, "2"'] & & \boxed{2}\otimes\boxed{3} \arrow[r, "2"] & \boxed{3}\otimes\boxed{2} \\ & & \boxed{1}\otimes\boxed{3} \arrow[ur, "1"'] & & \end{tikzcd}\right) \oplus \left(2\otimes 1 \xrightarrow{2} \boxed{3}\otimes\boxed{1} \xrightarrow{1} \boxed{3}\otimes\boxed{2} \right)$$

↘ iso to $B^{(3)}_{(2)}$ crystal

$$\begin{tikzcd} & & \boxed{2|2} \arrow[dr] & & \\ \boxed{1|1} \arrow[r] & \boxed{1|2} \arrow[ur] \arrow[dr] & & \boxed{2|3} \arrow[r] & \boxed{3|3} \\ & & \boxed{1|3} \arrow[ur] & & \end{tikzcd}$$

(4) More generally, we have crystals of (not necessarily row) tableaux

the Kashiwara operators will not be defined directly using tableaux. $\Leftarrow$ $B^{(n)}_{\lambda}$

$(n) \longrightarrow$ $n = r+1$, alphabet size

$\lambda \longrightarrow$ any partition, independent of n

$\downarrow$

GL(n) crystal whose elements, bijection onto a connected component of $\mathbb{B}^{\otimes|\lambda|}$, are SSYT of shape λ; weight record multiplicities as in the row case.

Instead. for $p \in SSYT(\lambda)$, $f_i(p) := RR^{-1} \circ f_i \circ \underline{RR(T)}$

We have the "row reading" map $\mathbb{B}^{\otimes|\lambda|}$

$$RR : B^{(n)}_{\lambda} \longrightarrow \mathbb{B}^{\otimes k}$$

, read bottom to top, left to right

where $\mathbb{B} = B^{(n)}$, $k = |\lambda|$.

descent, another row

e.g. $\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline 3 & 4 & \\ \hline \end{array} \longmapsto \boxed{3} \otimes \boxed{4} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2}$

Prop: RR is a morphism of crystals (by the way we defined f_i, really)

(5) Important Thm (3.2):

$RR(\mathbb{B}_\lambda^{(n)})$ is a connected component of $\mathbb{B}^{\otimes k}$, and has a unique highest wt elt, u_λ, the Yamanouchi tableaux with only i's in Row i $\forall i$.

Remarks: — The theorem helps decompose $\mathbb{B}^{\otimes k}$ (plethysm),

or, use Thm 4.11

— $\mathbb{B}$ is Stembridge by inspection, hence so is $\mathbb{B}^{\otimes k}$ by the tensor closure property of Stembridge crystals. So once we know $RR(\mathbb{B}_\lambda^{(n)})$ is connected, the uniqueness of the highest wt elt is automatic by Thm 4.12.

6.1. Column reading in type A.

↓

This is the embedding CR : $\mathcal{B}_\lambda^{(n)} \longrightarrow \mathbb{B}^{\otimes|\lambda|}$

↓

read columns from left to right, bottom to top in each column

e.g. $$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & \\ \cline{1-3} 4 & & & \\ \cline{1-1} \end{array} \longmapsto \boxed{4} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{2}$$

Results: (A) Thm 6.3 $CR(\mathcal{B}_\lambda)$ is a connected component of $\mathbb{B}^{\otimes k}$, $k = |\lambda|$.
It has a unique highest wt elt, namely $cr(u_\lambda)$. (same as RR)

(B) CR is an embedding of crystals.

c) Thm 6.1 $\mathcal{C}$ is a connected Stembridge $GL(n)$ crystal of h.w. $\lambda \Longrightarrow \mathcal{C} \cong B_\lambda$.

In particular, $RR(B_\lambda) \cong CR(B_\lambda)$.

Because Stembridge crystals are characterized by their highest wts. (Thm 4.13)

d) Cor 6.2. Any connected component $\mathcal{C}$ of $\mathbb{B}^{\otimes k}$ is isomorphic to B_λ for some $\lambda \vdash k$.

Pf: Need to show $hw(\mathcal{C}) = (\lambda_1, \dots, \lambda_n)$ is necessarily a partition, i.e., that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$; then use (c).

$\mathcal{C} \subset \mathbb{B}^{\otimes k} \Rightarrow \mathcal{C}$ is Stembridge $\Longrightarrow \mathcal{C}$ is h.w. $\xRightarrow{\text{Prop 2.16.}}$ $(\lambda_1, \dots, \lambda_n)$ is dominant $\Rightarrow \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ □

So, via the row reading (or column reading) embedding of tableaux we can obtain all connected components of $B^{\otimes k}$,

i.e., the map $RR : \{ B_\lambda^{(n)} : \lambda \vdash k \} \longrightarrow \{$ iso classes of connected components of $B^{\otimes k} \}$

is surjective.

essential, some component may only be isomorphic to a B_λ without being B_λ, see Fig 3.3.

(E). "Row and column readings do not exhaust all possible embeddings of B_λ into $B^{\otimes k}$. We'll see in Ch 8 that there's a nice way to understand all embeddings via plactic relations." (Page 83)

Question: How are the tensors in $RR(\mathcal{B}_\lambda)$ and $CR(\mathcal{B}_\lambda)$ related?

(when λ is hook shape they are identical)

Examples. (1). $\lambda = (2,2)$, $n = 2$, $r = 1$ $k = |\lambda| = 4$

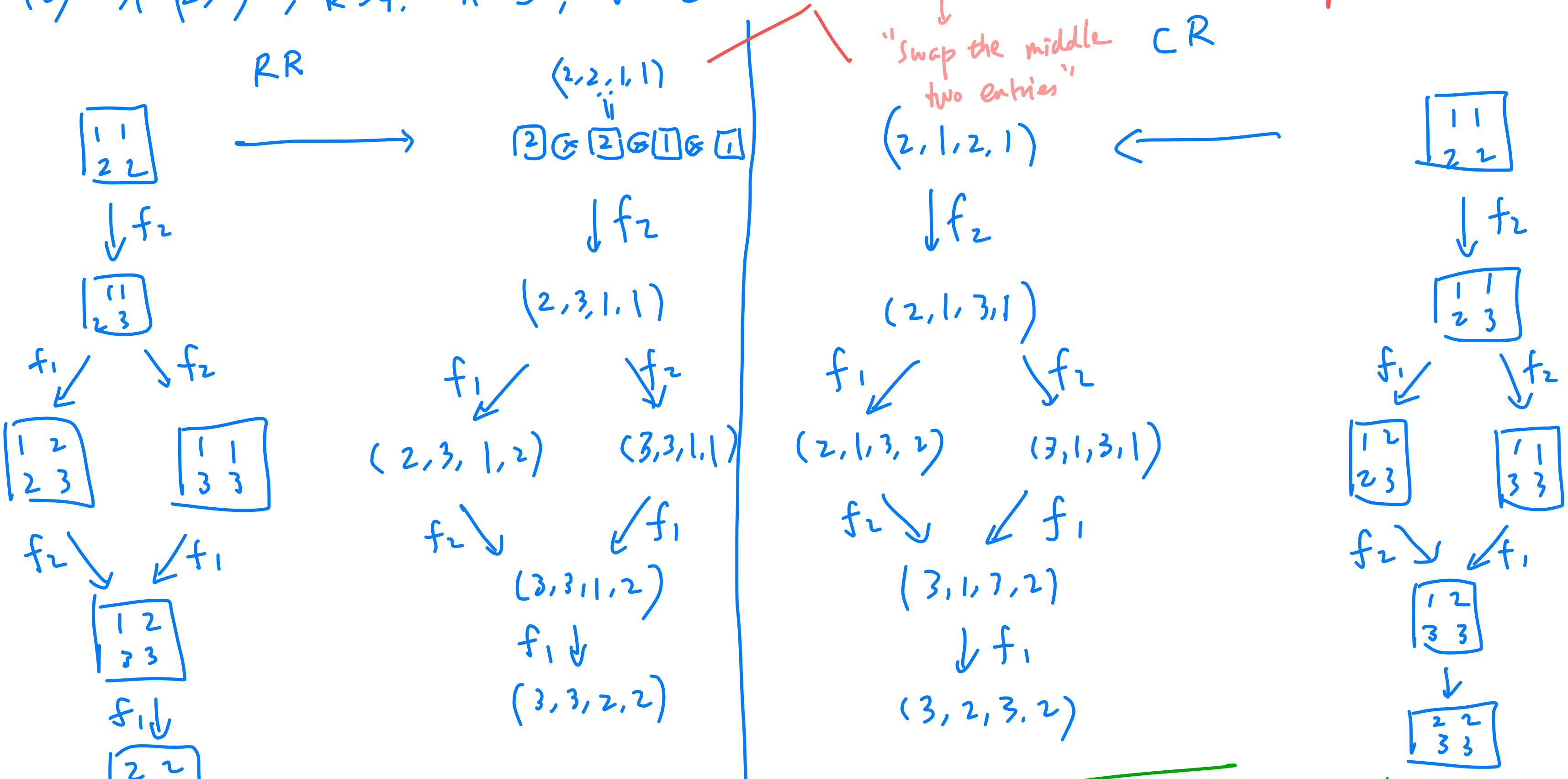
$$\mathbb{B}^{\otimes 4}$$

$$\begin{array}{|cc|}\hline 1 & 1 \\ 2 & 2 \\ \hline \end{array} \xrightarrow{RR} \boxed{2}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{1} \quad , \quad \boxed{2}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{1} \xleftarrow{CR} \begin{array}{|cc|}\hline 1 & 1 \\ 2 & 2 \\ \hline \end{array}$$

$\downarrow$ the only possible elt in $\mathcal{B}_\lambda$

(2) $\lambda=(2,2)$, $k=4$. $n=3$, $r=2$

How are these sides related? Plactic relations?

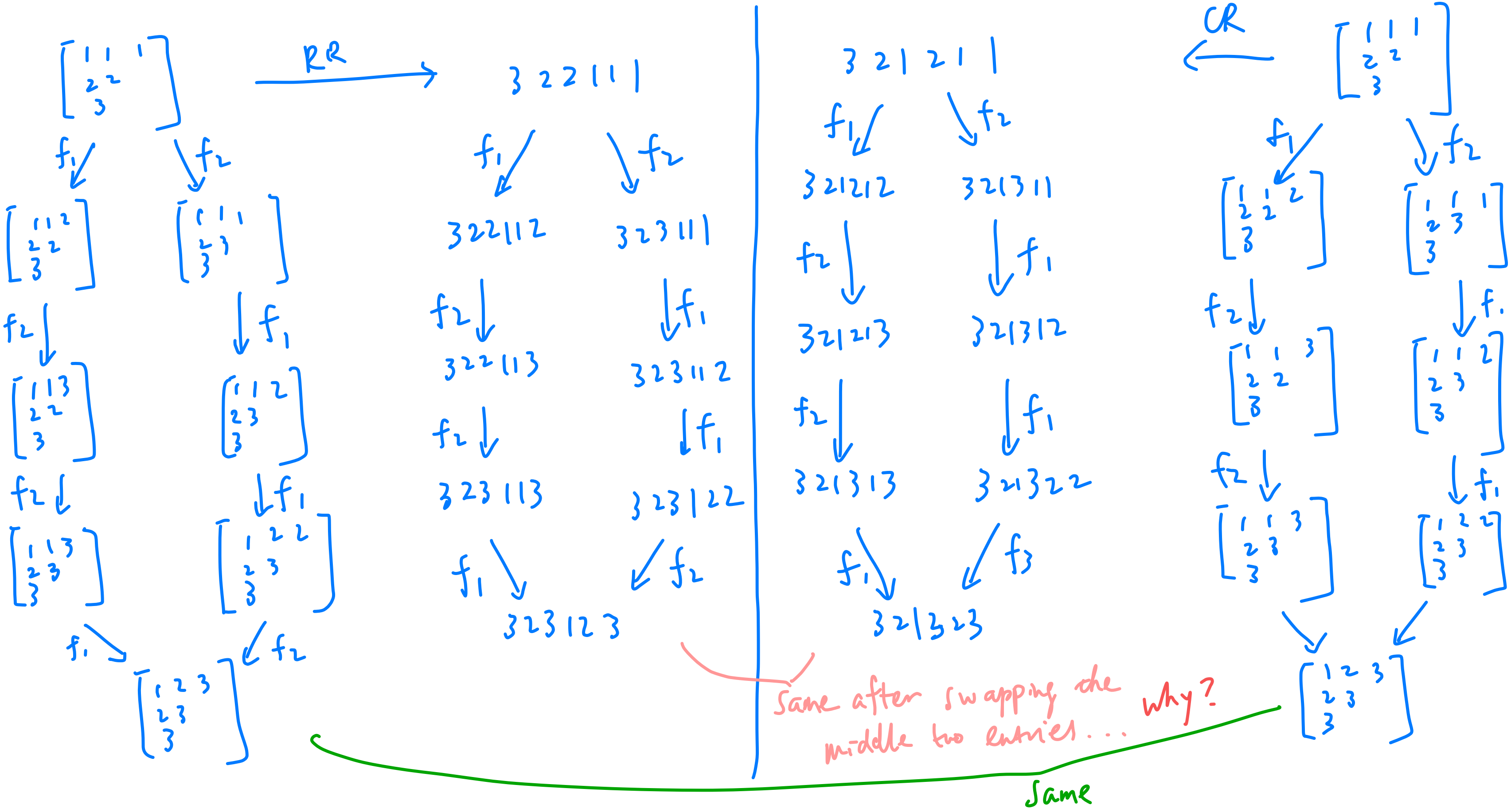


$RR^{-1}\circ f\circ RR = CR^{-1}\circ f\circ CR$

Same. obvious? → True but not trivial.

See Thm 7.3.6. of Hong-Kang.

13) $\lambda = (3,2,1)$, $k=6$, $n=3$, $r=2$



Same after swapping the middle two entries... why?

Same