Chapter 4: Stembridge Crystals

Problem 4.2: Give each dom. weight λ a Seminormal Crystal w/ highest weight λ, also want:

- these crystals are closed under $\otimes$ [up to irred. decomp.]
- the character* of a crystal matches λ.

Seminormal Crystals are "too big" [e.g. pinched crystal] for this

Also, apparently not all (see Stembridge '03) Kashiwara crystals correspond to representations (?), these ones will

We will define Stembridge Crystals to resolve this in the Simply laced Case. It appears this concept may extend to other types in a less complete way (?)

??

Approach: look at Crystal restricted to 2 simple roots, α_i & α_j, becomes either $\underbrace{A_1 \times A_1}_{\bullet \quad \bullet}$ or $\underbrace{A_2}_{\bullet\!-\!\bullet}$ crystal

$\langle \alpha_i, \alpha_j^\vee \rangle = 0$ $\qquad$ $\langle \alpha_i, \alpha_j^\vee \rangle = -1$

Because simply laced

We impose Axioms/conditions that make these Crystals more "navigable."

In Simply laced crystals:

Lem 4.4: If $\langle \alpha_i, \alpha_j^\vee \rangle \neq 0$, $\varepsilon_i(x) > 0$, $\varphi_j(e_i x) - \varphi_j(x) = \varepsilon_j(e_i x) - \varepsilon_j(x) + 1$

Knew: • $\varphi_j(x) - \varepsilon_j(x) = \langle wt(x), \alpha_j^\vee \rangle$

• $\varphi_j(e_i x) - \varepsilon_j(e_i x) = \langle wt(e_i x), \alpha_j^\vee \rangle$

$= \langle wt(x) + \alpha_i, \alpha_j^\vee \rangle$

$= \langle wt(x), \alpha_j^\vee \rangle + \langle \alpha_i, \alpha_j^\vee \rangle$

we want more!

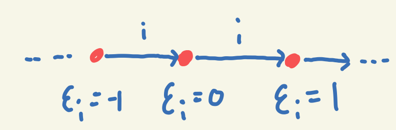
Stembridge Axioms (assume Φ is simply laced)

S0: if $e_i(x)=0$, then $\varepsilon_i(x)=0$

implications: • if the i root string through x isn't infinite in the upward / e_i direction, then

$$\varepsilon_i(x)=\max(k: e_i^k(x)\neq 0)$$

• if $e_i(x)=0$, $\varphi_i(x)=\langle wt(x), \alpha_i^\vee(x)\rangle$



e_i cannot send any of these to 0 as $\varepsilon_i<0$, decreasing w/ each application of e_i

S0': If $f_i(x)=0$, then $\varphi_i(x)=0$. Dual to S0.

S1: For $i,j\in I$, $x\in \mathcal{C}$ with $\varepsilon_i(x)>0$

- $\varepsilon_j(e_ix)=\varepsilon_j(x)$, or
- $\varepsilon_j(e_ix)=\varepsilon_j(x)+1$ and α_i,α_j aren't orthogonal

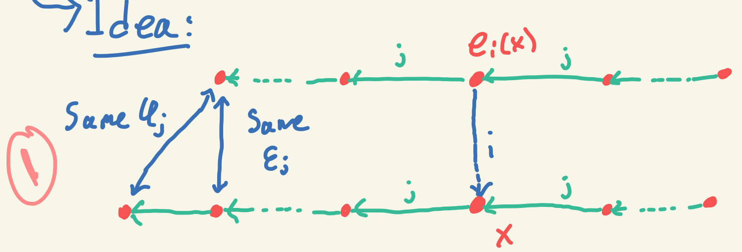
Implications: one of the following holds (Prop 4.5):

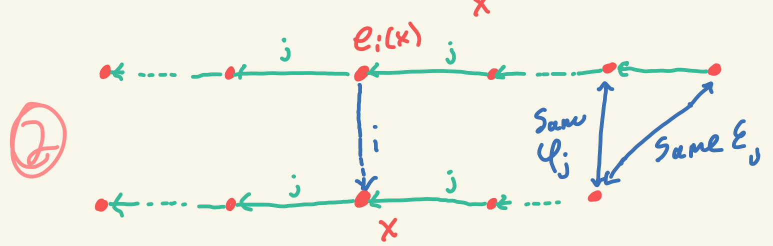
1) $\langle\alpha_i,\alpha_j^\vee\rangle=-1$, $\varepsilon_j(e_ix)=\varepsilon_j(x)$, $\varphi_j(e_ix)=\varphi_j(x)-1$
2) " ", $\varepsilon_j(e_ix)=\varepsilon_j(x)+1$, $\varphi_j(e_ix)=\varphi_j(x)$
3) $\langle\alpha_i,\alpha_j\rangle=0$, $\varepsilon_j(e_ix)=\varepsilon_j(x)$, $\varphi_j(e_ix)=\varphi_j(x)$

root strings "shift" when α_i,α_j are not orthogonal

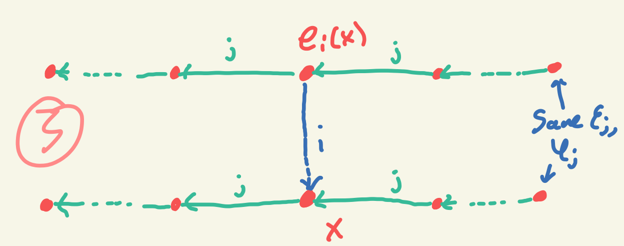
Idea:

α_i,α_j not orth, $\langle\alpha_i,\alpha_j^\vee\rangle=-1$





α_i,α_j orth.



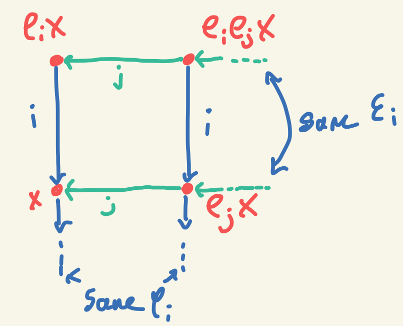
If $\mathcal{C}$ is Seminormal:

- α_i,α_j orthogonal $\Rightarrow$ e_i is "isomorphism" of j-strings when nonzero
- α_i,α_j not orthogonal $\Rightarrow$ e_i "shifts" j-strings by 1.

S2: For $i,j\in I$ distinct, $x\in\mathcal{C}$ w/ $\varepsilon_i(x)>0$, $\varepsilon_j(e_i x)=\varepsilon_j(x)>0$

[cases 1 & 3 above in the "j-positive" region]

Then: $e_ie_jx = e_je_ix$, and $\varphi_i(e_jx)=\varphi_i(x)$



Implication: if $\varepsilon_j(e_ix)=\varepsilon_j(x)$ or $\varepsilon_i(e_jx)=\varepsilon_i(x)$

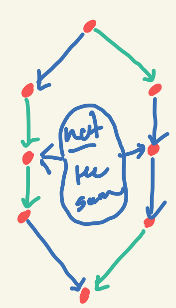
then $e_ie_jx=e_je_ix$ (see fig. 4.4)

S3: If $i,j\in I$, $x\in\mathcal{C}$ w/ $\varepsilon_j(e_ix)=\varepsilon_j(x)+1>1$

[case 2 above, for i & j] Then: $e_je_i^2e_jx = e_ie_j^2e_ix \neq 0$

and $\varphi_i(e_jx)=\varphi_i(e_j^2e_ix)$, $\varphi_j(e_ix)=\varphi_j(e_i^2e_jx)$.

Implication: lem 4.6: in S3,



$\boxed{\text{See also: Fig 3 in Stembridge '03}}$

Also: "Dual statements S1', S2', S3'

Def: A crystal $\mathcal{C}$ is stembridge if $\mathcal{C}$ is seminormal and satisfies S1, S2, S3, S1', S2', S3'.

$\Downarrow$

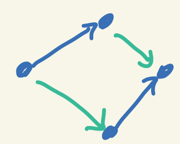
" " weak stembridge if $\mathcal{C}$ satisfies S0, S0', S1, S2, S3, S1', S2', S3'.

If $\mathcal{C}$ seminormal: if e_i doesn't shift the top when moving between j-strings, then e_i & e_j commute at that point, and e_j doesn't shift bottom

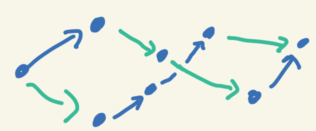
If $\mathcal{C}$ seminormal:

Summary: Stembridge crystals ~~restricted to~~ Levi branched to roots α_i, α_j have connected components that "look like"

- Arrows in i shift j strings by at most 1, vice versa
- rectangles if $\langle \alpha_i, \alpha_j^\vee \rangle = 0$



- "twisted rectangles" [locally] if $\langle \alpha_i, \alpha_j^\vee \rangle \neq 0$



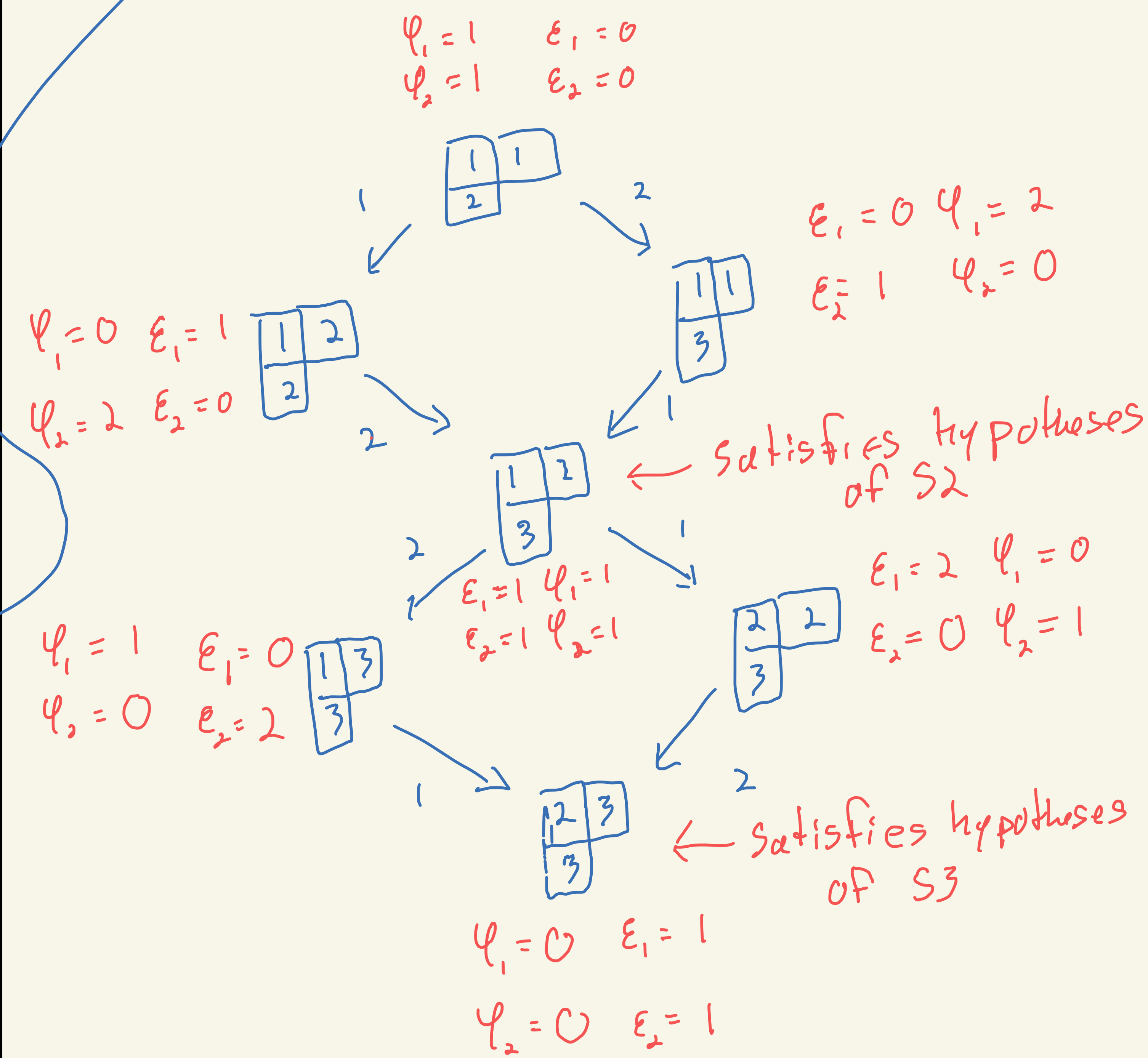
Weak Stembridge crystals are like this on the seminormal-like part of the crystal.

More to think about/Questions:

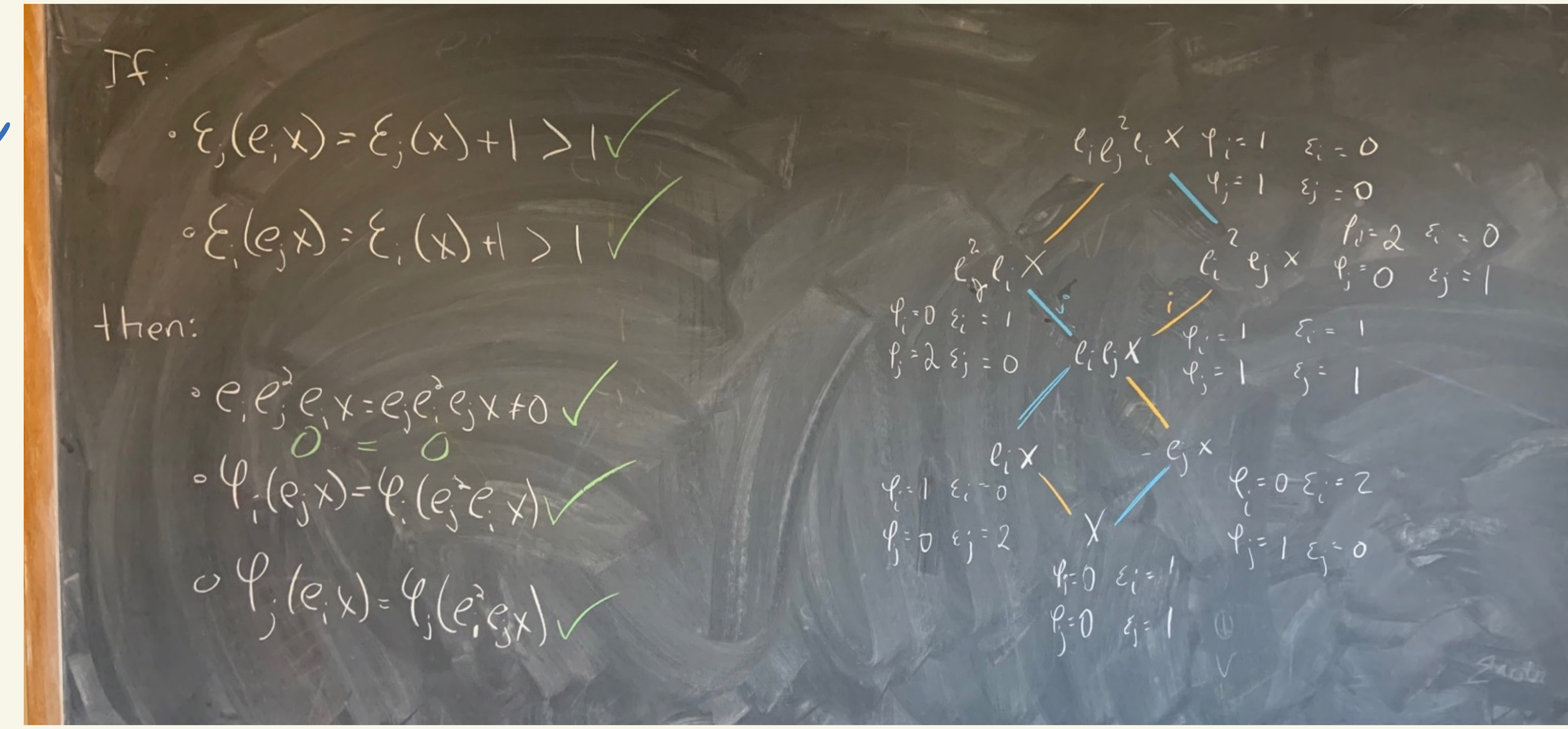
- How is Crystal in Fig 3.1 (p.32) not stembridge? Ex 4.1
- P.40: why must all A_2 crystals lie in B_λ for some λ?
- How does S1 => S1'? → Lemma 4.4
- Props 4.7, 4.9: important? [Not in this chapter!]
- Character ex. in Sec. 4.1?

More to do: - Read Stembridge '03 paper? [Notation is different enough to be a bit confusing]

Why isn't this Stembridge?

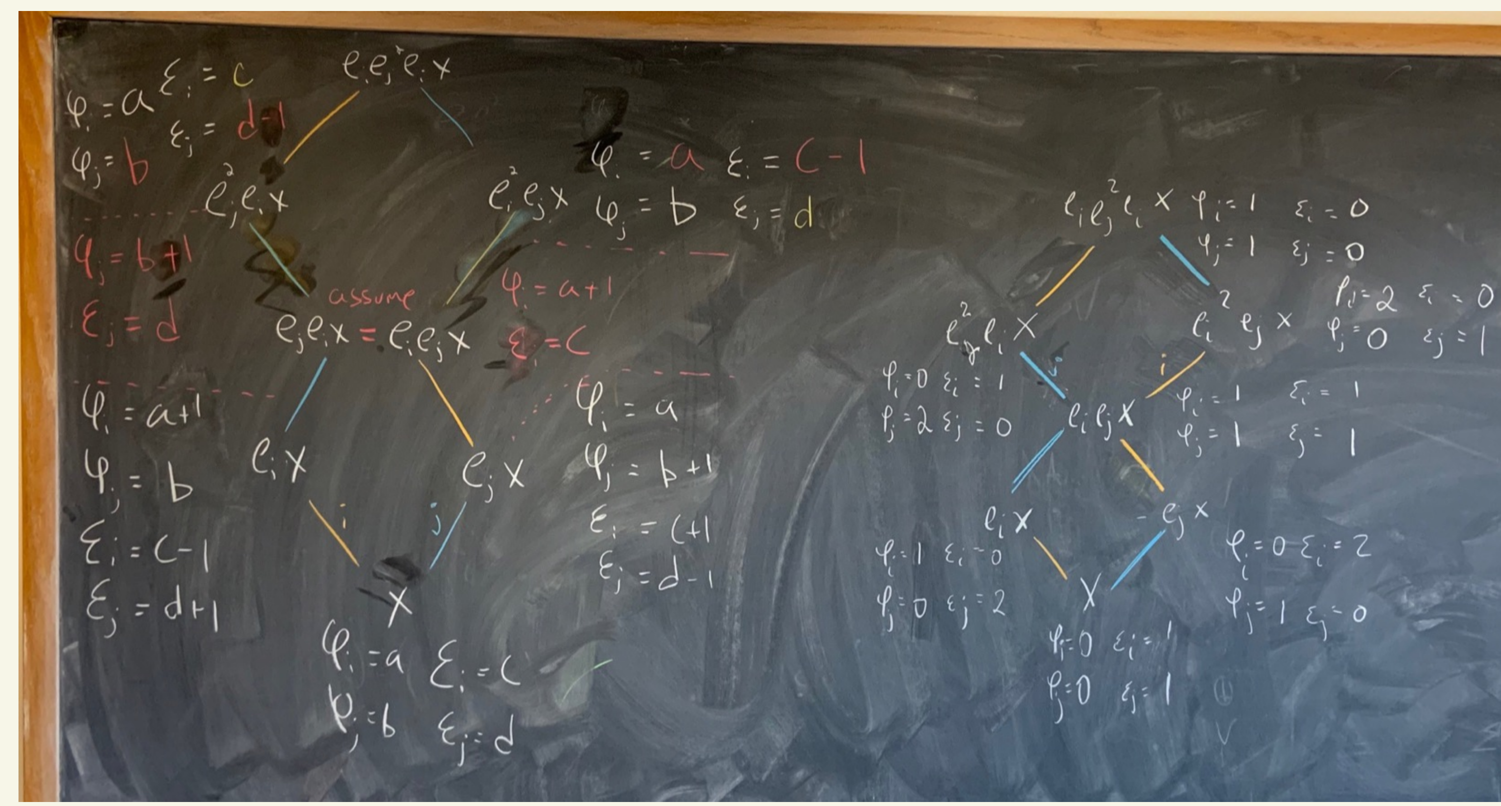


For Friday: - Finish Ex 4.1

The crystal 

fails to satisfy axiom (S2), does satisfy (S3)

-Typo in Lemma 4.4 -> also requires assuming S2



- Also: Hard to verify st. axioms for crystals of tableaux generally,

But they hold for crystals in Figs. 3.2, 3.3(left), 2.2