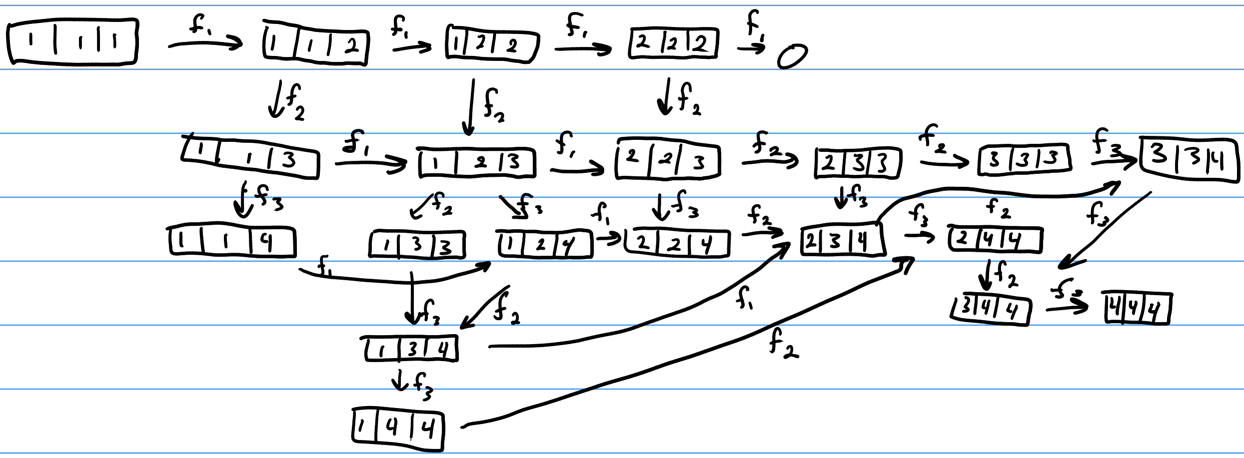
Row Crystals $\boxed{x_1 \mid x_2 \mid x_3}$

$x_1 \le x_2 \le x_3$, $x_i \in \{1,2,3,4\}$ ↙ not # of boxes.

f_1 bumps a $1 \to 2$ if possible.



Check the axioms: $\mathrm{wt}(\boxed{x_1 \mid x_2 \mid x_3}) = (\mu_1, \mu_2, \mu_3, \mu_4)$ where $\mu_i := \#$ of i's in $\{x_1, x_2, x_3\}$.

$\mathrm{wt}(y) = \mathrm{wt}(f_i(x)) + \alpha_i$. String length maps are obvious.

$\varphi_i = \#$ of i's, $\varepsilon_i = \#$ of $(i+1)$'s.

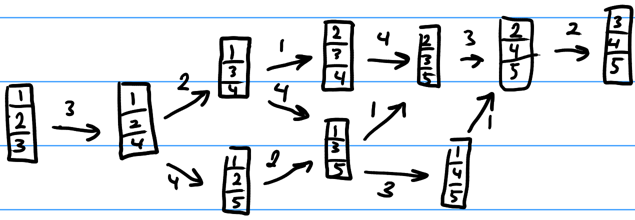
$$\varphi_i = \varepsilon_i + \langle \mathrm{wt}(x), \alpha_i^\vee \rangle$$
$$\langle \mathrm{wt}(x), e_i - e_{i+1} \rangle.$$

Example 2.26: Crystal of Columns. 4 labels $n=4$.

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \xrightarrow{3} \begin{bmatrix}1\\2\\4\end{bmatrix} \xrightarrow{2} \begin{bmatrix}1\\3\\4\end{bmatrix} \xrightarrow{1} \begin{bmatrix}2\\3\\4\end{bmatrix}$$

$\varphi_1 = \varphi_2 = 0$,
$\varphi_3 = 1$.

$n=5$



Question: If $n=6$, do we get a dual picture to example 2.25 with $n=4$?

Axioms: $\varphi_i(x) = 1$ if x has an i, but no $i+1$. $\varepsilon_i = 1$ if x has i but no $i-1$.

(A1) wt map is the same as in the crystal of rows.

$\varphi_i(y) = \varphi_i(f_i(y)) + 1$ ✓ $\mathrm{wt}(y) = \mathrm{wt}(f_i(y)) + \alpha_i = \mathrm{wt}(f_i(y)) + (\varepsilon_i - \varepsilon_{i+1})$

(A2) $\varphi_i(x) = \varepsilon_i(x) + \langle \mathrm{wt}(x), \alpha_i^\vee \rangle$

Example 2.30: $\mathbb{B}\otimes\mathbb{B}$, for $\mathbb{B}$ the $GL(3)$ crystal

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

$$wt(\boxed{i}) = e_i$$

$\mathbb{B}\otimes\mathbb{B}$:

In $\mathbb{B}$, $\varphi_i(\boxed{j}) = \delta_{i,j}$, $\varepsilon_i(\boxed{j}) = \delta_{i,j-1}$

$\boxed{1}$	$\boxed{2}$	$\boxed{3}$
$\varphi_1 = 1$	$\varphi_1 = 0$	$\varphi_1 = 0$
$\varphi_2 = 0$	$\varphi_2 = 1$	$\varphi_2 = 0$
$\varepsilon_1 = 0$	$\varepsilon_1 = 1$	$\varepsilon_1 = 0$
$\varepsilon_2 = 0$	$\varepsilon_2 = 0$	$\varepsilon_2 = 1$

$$\begin{tikzcd}
\boxed{1}\otimes\boxed{1} \arrow[r, "1"] & \boxed{1}\otimes\boxed{2} \arrow[r, "2"] \arrow[d, "1"] & \boxed{1}\otimes\boxed{3} \arrow[d, "1"] \\
\boxed{2}\otimes\boxed{1} \arrow[d, "2"] & \boxed{2}\otimes\boxed{2} \arrow[r, "2"] & \boxed{2}\otimes\boxed{3} \arrow[d, "2"] \\
\boxed{3}\otimes\boxed{1} \arrow[r, "1"] & \boxed{3}\otimes\boxed{2} & \boxed{3}\otimes\boxed{3}
\end{tikzcd}$$

2 connected components

$$f_i(\boxed{x}\otimes\boxed{y}) = \begin{cases} f_i(\boxed{x})\otimes\boxed{y} & \text{if } \delta_{i,y} \leq \delta_{i,x-1} \\ x\otimes f_i(\boxed{y}) & \text{if } \delta_{i,y} > \delta_{i,x-1} \end{cases}$$

$\square\otimes\square\otimes\square$

Type A_r

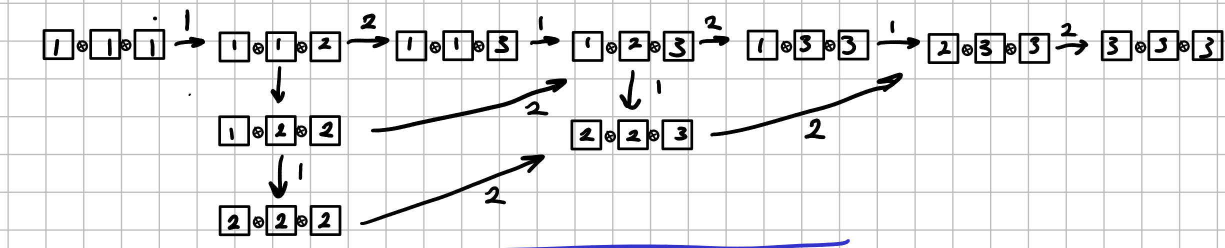
$\mathbb{B}\otimes\mathbb{B}\otimes\mathbb{B}$ for $\mathbb{B}$ the standard $GL(3)$-crystal.

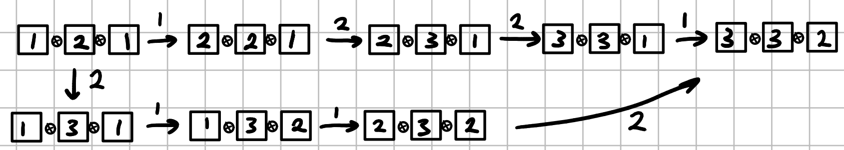
$$
\begin{tikzcd}
\boxed{1}\otimes\boxed{1}\otimes\boxed{1} \arrow[r, "1"] & \boxed{1}\otimes\boxed{1}\otimes\boxed{2} \arrow[r, "2"] \arrow[d, "1"] & \boxed{1}\otimes\boxed{1}\otimes\boxed{3} \arrow[r, "1"] & \boxed{1}\otimes\boxed{2}\otimes\boxed{3} \arrow[r, "2"] \arrow[d, "1"] & \boxed{1}\otimes\boxed{3}\otimes\boxed{3} \arrow[r, "1"] & \boxed{2}\otimes\boxed{3}\otimes\boxed{3} \arrow[r, "2"] & \boxed{3}\otimes\boxed{3}\otimes\boxed{3} \\
& \boxed{1}\otimes\boxed{2}\otimes\boxed{2} \arrow[urr, "2"] \arrow[d, "1"] & & \boxed{2}\otimes\boxed{2}\otimes\boxed{3} \arrow[urr, "2"] & & & \\
& \boxed{2}\otimes\boxed{2}\otimes\boxed{2} \arrow[urr, "2"] & & & & &
\end{tikzcd}
$$

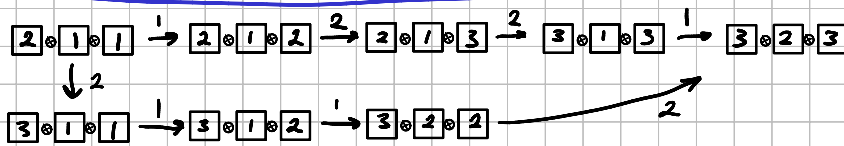
$$
\begin{tikzcd}
\boxed{1}\otimes\boxed{2}\otimes\boxed{1} \arrow[r, "1"] \arrow[d, "2"] & \boxed{2}\otimes\boxed{2}\otimes\boxed{1} \arrow[r, "2"] & \boxed{2}\otimes\boxed{3}\otimes\boxed{1} \arrow[r, "2"] & \boxed{3}\otimes\boxed{3}\otimes\boxed{1} \arrow[r, "1"] & \boxed{3}\otimes\boxed{3}\otimes\boxed{2} \\
\boxed{1}\otimes\boxed{3}\otimes\boxed{1} \arrow[r, "1"] & \boxed{1}\otimes\boxed{3}\otimes\boxed{2} \arrow[r, "1"] & \boxed{2}\otimes\boxed{3}\otimes\boxed{2} \arrow[urr, "2"] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{2}\otimes\boxed{1}\otimes\boxed{1} \arrow[r, "1"] \arrow[d, "2"] & \boxed{2}\otimes\boxed{1}\otimes\boxed{2} \arrow[r, "2"] & \boxed{2}\otimes\boxed{1}\otimes\boxed{3} \arrow[r, "2"] & \boxed{3}\otimes\boxed{1}\otimes\boxed{3} \arrow[r, "1"] & \boxed{3}\otimes\boxed{2}\otimes\boxed{3} \\
\boxed{3}\otimes\boxed{1}\otimes\boxed{1} \arrow[r, "1"] & \boxed{3}\otimes\boxed{1}\otimes\boxed{2} \arrow[r, "1"] & \boxed{3}\otimes\boxed{2}\otimes\boxed{2} \arrow[urr, "2"] & &
\end{tikzcd}
$$

$$\boxed{3}\otimes\boxed{2}\otimes\boxed{1}$$







$\boxed{3}\otimes\boxed{2}\otimes\boxed{1}$

Chapter 3 - Crystals of Tableaux

Theorem 3.2: Let λ a partition of k with length $\leq n$.
Then $RR(\mathcal{B}_\lambda)$ is a connected component of $\mathbb{B}^{\otimes k}$
and has a unique highest weight element, $RR(u_\lambda)$ where u_λ is the Yamanouchi Tableau

Proof that $RR(u_\lambda)$ is unique highest weight element of $RR(\mathcal{B}_\lambda)$.

$$u_\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}.$$

If $RR(T)$ is a highest weight element, $e_i(RR(T))=0$, hence $\varepsilon_i RR(T)=0$.

$$\varepsilon_i(x_k \otimes \cdots \otimes x_1) = \max_{j=1}^{k}\left(\sum_{h=1}^{j}\varepsilon_i(x_h) - \sum_{h=1}^{j-1}\varphi_i(x_k)\right)$$

$$RR(T) = RR(R_\ell)\otimes\cdots\otimes RR(R_1)$$

Recall $\varepsilon_i(\boxed{}) = \#(i+1)$'s,
$\varphi_i(\boxed{}) = \#\ i$'s.

Forces the highest weight element to be u_λ.

Exercise 3.1 $\mathbb{B}$ = standard $GL(n)$ crystal
Show that for fixed k, # of full connected subcrystals of $\mathbb{B}^{\otimes k}$ is independent of n.

Problems to do/check using SAGE.
Look @ the SAGE thematic tutorial
Try problem 2.4, 2.8, 2.10 using tableaux, 3.2

Note: Consider reading or skimming §4.2 before §4.1.