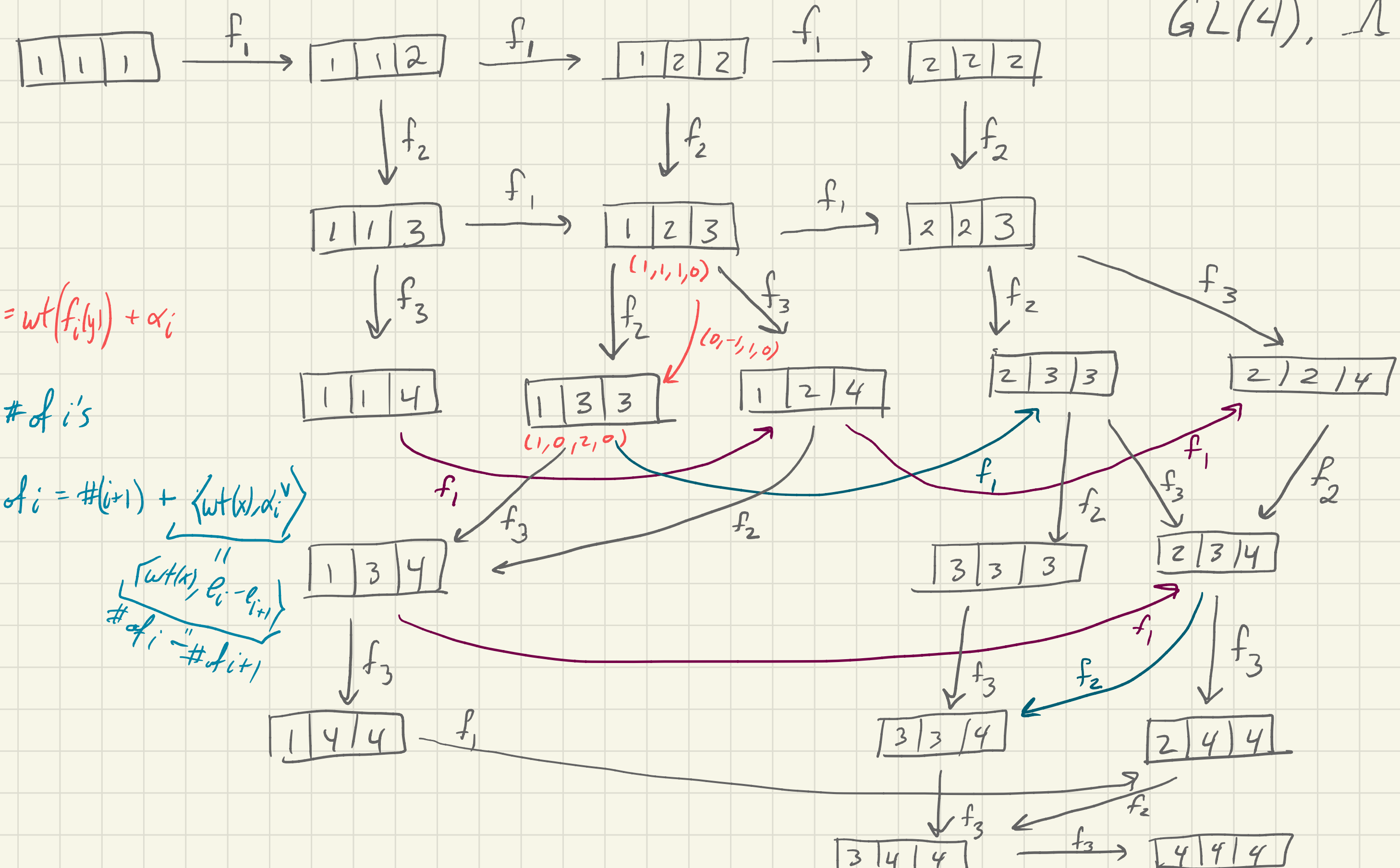
February 11, 2020

Goals:
- Row & Column crystals (Ex 2.25, 2.26)
- Tensor product crystal (Ex 2.30)
- Signature Rule examples (Ex 2.34)

Row Crystals: $\boxed{x_1 \mid x_2 \mid x_3}$

$x_1 \le x_2 \le x_3$, $x_1, x_2, x_3 \in \{1,2,3,4\}$ ← not the # of boxes

$GL(4)$, $\Lambda = \mathbb{Z}^4$



$wt(y) = wt(f_i(y)) + \alpha_i$

$\varphi_i(x) = \#$ of i's

A2) $\#$ of $i = \#(i+1) + \underbrace{\langle wt(x), \alpha_i^\vee \rangle}_{\substack{\| \\ \langle wt(x),\, e_i - e_{i+1} \rangle \\ \#\text{ of } i - \#\text{ of } i+1}}$

Example 2.26.

$n=3$.

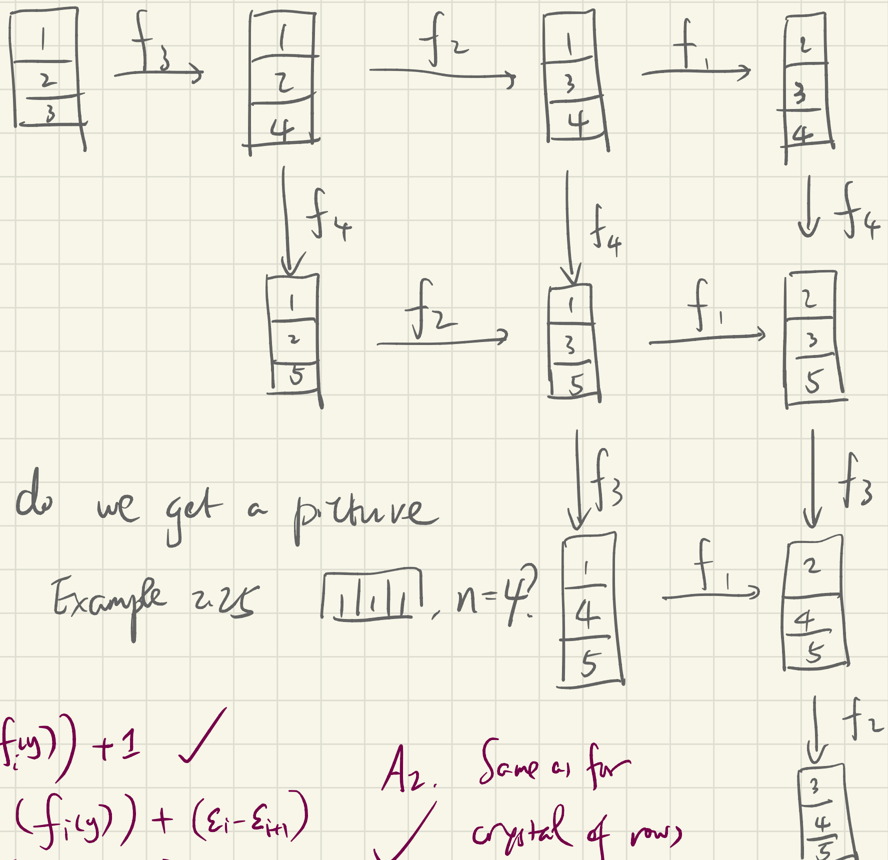
$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}$$

$\varphi_1=\varphi_2=\varphi_3=0$

$n=4$

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array} \xrightarrow{f_3} \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 4\\ \hline\end{array} \xrightarrow{f_2} \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline 4\\ \hline\end{array} \xrightarrow{f_1} \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 4\\ \hline\end{array}$$

Example 2.26. $n=5$.



Q: If $n=6$, do we get a picture 'dual' to Example 2.25 $\boxed{1|1|1|1}$, $n=4$?

The axioms:

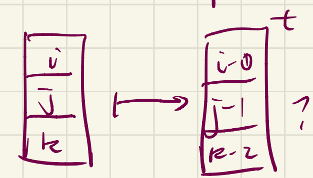
A1. $\varphi_i(y) = \varphi_i(f_i y) + 1$ ✓

$wt(y) = wt(f_i(y)) + (\varepsilon_i - \varepsilon_{i+1})$

$(\ldots, \underset{i}{1}, \underset{i+1}{0}, \ldots)$ $(\ldots 0, 1 \ldots)$

A2. Same as for crystal of rows ✓

Q: Is this dual to the row tableaux example in Fig 2.2 ($\boxed{1|1|1|1}$, $n=3$) via the map



?

$GL(3)$ $\quad B = \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$

$\varphi_1 = 1$ $\varepsilon_1 = 0$	$\varphi_1 = 0$ $\varepsilon_1 = 1$	$\varphi_1 = 0$ $\varepsilon_1 = 0$
$\varphi_2 = 0$ $\varepsilon_2 = 0$	$\varphi_2 = 1$ $\varepsilon_2 = 0$	$\varphi_2 = 0$ $\varepsilon_2 = 1$

$B \otimes B$

$$\begin{tikzcd}
\boxed{1}\otimes\boxed{1} \arrow[r, "f_1"] & \boxed{1}\otimes\boxed{2} \arrow[r, "f_2"] \arrow[d, "f_1"] & \boxed{1}\otimes\boxed{3} \arrow[d, "f_1"] \\
\boxed{2}\otimes\boxed{1} \arrow[d, "f_2"] & \boxed{2}\otimes\boxed{2} \arrow[r, "f_2"] & \boxed{2}\otimes\boxed{3} \arrow[d, "f_2"] \\
\boxed{3}\otimes\boxed{1} \arrow[r, "f_1"] & \boxed{3}\otimes\boxed{2} & \boxed{3}\otimes\boxed{3}
\end{tikzcd}$$

$\boxed{j} \otimes \boxed{K}$: $\; j > K$

$$f_i\left(\boxed{j} \otimes \boxed{K}\right) = \begin{cases} f_i(\boxed{j}) \otimes \boxed{K} & \varphi_i(K) \le \varepsilon_i(j) \iff \delta_{iK} \le \delta_{i,j-1} \\ \boxed{j} \otimes \underbrace{f_i(K)}_{=\,\boxed{K+1}} & \delta_{iK} > \delta_{i,j-1} \iff i = K,\ i \ne j-1 \end{cases}$$

$$\varphi_i(\boxed{K}) > \varepsilon_i(\boxed{j})$$

Q: What intuition can we find for $\otimes$ f_i's and e_i's?