I should:

- Do the B_ℓ algebra computations ($so_{2\ell+1}$)

- Point out the sl_2-triples.

- Talk about the sl_2 reps motivate axiom A_2.
  { wt x vs wt($e \cdot x$)
  string length

- Do the B_ℓ $\curvearrowright V$ natural rep; motivate A_1.

Hints of crystal axioms from representation theory of Lie algebras

(Technically, crystals come from reps of the quantum gps $U_q(\mathfrak{g})$ associated to Lie algebras $\mathfrak{g}$. Why $U_q(\mathfrak{g})$ rather than $\mathfrak{g}$? — we may want to understand this later, or not.)

Lie algebra facts from last time: (1) Any semisimple Lie algebra L has a maximal toral subalgebra H which acts 'semisimply'/'diagonally' on every finite dimensional rep V of L in the sense that

$[H,H]=0$

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda \quad \text{where } \Lambda \subseteq H^*,\ V_\lambda \neq 0\ \forall \lambda \in \Lambda,$$

and $V_\lambda = \{ v \in V : h \cdot v = \lambda(h) \cdot v \quad \forall h \in H \}$.

(2) The elts of Λ are called weights. For the special adjoint action of L on itself, the weights are called roots. The roots always forms an abstract root system.

<u>Example:</u> $sl_{n+1} := sl_n(\mathbb{C})$ realizes the root system of type A_n.

e.g. $L = sl_4 \rightsquigarrow H =$ diagonal matrices in gl_4.

A basis for L: $\underbrace{\{e_{11}-e_{22}, e_{22}-e_{33}, e_{33}-e_{44}\}}_{\text{basis of } H} \cup \{e_{ij} : 1 \le i,j \le 4, i \ne j\}$

$\forall h = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{bmatrix} \in H, \quad 1 \le i,j \le 4, \quad i \ne j,$

$[h, e_{ij}] = he_{ij} - e_{ij}h = (a_i - a_j)\, e_{ij} = (\varepsilon_i - \varepsilon_j)(h)\, e_{ij}$

where ε_k is the functional in H^* w/ $\varepsilon_k(h) = a_k$.

It follows that $L = H \oplus \bigoplus_{i \neq j} \underline{L_{\varepsilon_i - \varepsilon_j}}$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \quad \mathrm{Span}(e_{ij})$

the root system: $\Phi = \{ \varepsilon_i - \varepsilon_j \mid i \neq j \}$.

a base Δ of Φ: $\{ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4 \}$

the positive roots w.r.t. Δ: $\Phi^+ = \Phi \cap \mathbb{N}\Phi = \{ \varepsilon_i - \varepsilon_j : i < j \}$.

all true for general L:

$\forall \alpha \in \Phi^+$, $\dim L_\alpha = 1$, and $\exists\, e_\alpha \in L_\alpha$, $f_\alpha \in L_{-\alpha}$

s.t. $(e_\alpha, f_\alpha, h_\alpha := [e_\alpha, f_\alpha])$ forms an $\mathfrak{sl}_2$-triple $\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$,

i.e., satisfies $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, f_\alpha] = -2f_\alpha$.

The set $\{ e_\alpha, f_\alpha : \alpha \in \Delta \}$ generate L (as a Lie algebra, via bracketing).

e.g.: sl_2-triples in sl_4

$(e_{13}, e_{31}, e_{11}-e_{33}) \leftarrow$ gen. by $\{e_{12}, e_{21}, e_{11}-e_{22}\}$ $\leftrightarrow$ simple roots

$\{e_{23}, e_{32}, e_{22}-e_{33}\}$

$\updownarrow$

not a simple root

Point: L is always made up of subalgebras iso to sl_2, for each sl_2 triple sl_α $(\alpha \in \Phi)$ any rep $L \curvearrowright V$, we get $sl_2 \hookrightarrow L \curvearrowright V$; to understand V it suffices to keep track of the sl_2-triples' actions.

Two computations.

(1). One more type : B_ℓ $(\ell \geq 1)$

The algebra: Take an ℓ-dim $\mathbb{C}$-vector space V. Fix a basis β. Take

$$L = \{ x \in gl(V) : f(x \cdot v, w) = -f(v, x \cdot v) \ \forall v, w \in V \}$$

where f is the bilinear form given by the Gram matrix $\underbrace{}_{(2\ell+1) \times (2\ell+1)}$

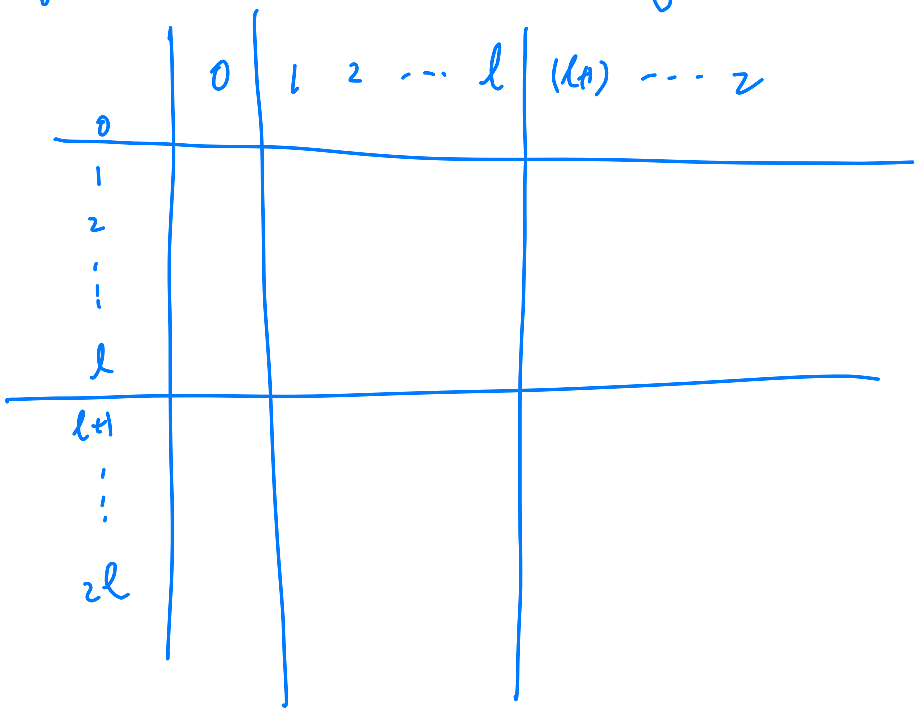
$$S = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & I_\ell \\ \hline 0 & I_\ell & 0 \end{array}\right]$$

Ex: (1). $f(v, w) = v^t S w$ (2) $x \in L \iff [x]_\beta^t S = -S [x]_\beta$

(3) $x \in L \iff x = \begin{bmatrix} 0 & -c^t & -b^t \\ b & m & p \\ c & q & -m^t \end{bmatrix}$ where $p = -p^t$, $q = -q^t$, $b, c, m \in gl_n$

In L, the set H of diagonal matrices forms a Cartan subalgebra.

Labelling the rows and columns of all matrices in L by



We get a natural basis of L where each elt happens to be a simultaneous eigenvector for the actions of all elts of H. i.e., a weight vector. The basis elts and their corresponding roots are:

(all indices i, j are from $\{1, 2, \ldots, l\}$; $\varepsilon_i \in H^*$ is as in Type A.)

conj conj

name	b_i	c_i	m_{ij} $(i \neq j)$	p_{ij} $(i<j)$	q_{ij}	h_i
formula	$e_{i,0} - e_{0,l+i}$	$e_{0,i} - e_{l+i,0}$	$e_{ij} - e_{l+j,l+i}$	$e_{i,l+j} - e_{j,l+i}$	$e_{l+j,i} - e_{l+i,j}$	$e_{ii} - e_{l+i,l+i}$
wt	ε_i	$-\varepsilon_i$	$\varepsilon_i - \varepsilon_j$	$\varepsilon_i + \varepsilon_j$	$-(\varepsilon_i + \varepsilon_j)$	NA / 0

The root system is $\Phi = \{\pm\varepsilon_i, \pm(\varepsilon_i - \varepsilon_j), \pm(\varepsilon_i + \varepsilon_j)\}$.

a base is $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_l\}$,

the corresponding sl_2-triples are

$$\{(m_{ij}, m_{ji}, h_{ij} := [m_{ij}, m_{ji}] = (e_{ii} - e_{l+i,l+i}) - (e_{jj} - e_{l+j,l+j})) : j = i+1\}$$

$$\cup \{(b_l, c_l, h_l := [b_l, c_l] = e_{ll} - e_{2l,2l})\};$$

these triples generate L.

(2). Finite dimensional representations of sl_2. (over $\mathbb{C}$).

Thm (Thm 8.5 in Erdmann-Wildon) For each $d \in \mathbb{Z}_{\geq 0}$ $\exists$ a unique sl_2-module V_d which has dimension $d+1$. V_d has a basis

$$v_{-d} \; \cdots \; v_i \; - \; - \; - \; v_{d-2} \; v_d$$

where $[h, v_i] = i \cdot v_i$ and all the nonzero actions of e, f are captured in the graph

$$v_{-d} \underset{f}{\overset{e}{\rightleftarrows}} v_{-d+2} \; \cdots \; v_i \; \cdots \; v_{d-2} \underset{f}{\overset{e}{\rightleftarrows}} v_d .$$

Note: ① The fact that the h-eigenvalues of v_i and ev_i / fv_i differ by 2 is no coincident: $h e v_i = e h v_i + [h, e] v_i$

$$= e\, i v_i + 2 e v_i = (i+2) e v_i$$

② For each i, the number $\varphi(v_i)$ of 'steps' to the left of v_i and the number $\varepsilon(v_i)$ of 'steps' to the right of v_i satisfy

$$\underbrace{\varphi(v_i)}_{\frac{i+d}{2}} = i + \underbrace{\varepsilon(v_i)}_{\frac{d-i}{2}} \longrightarrow \text{root string length equation.}$$

We can now see hints of all the crystal axioms:

B : encodes an eigenbasis for the H-action of a $\mathfrak{g}$-module;

e_i, f_i $(i \in I \longleftrightarrow \Phi)$: encode the actions of the $\mathfrak{sl}_2$-triples.

why not Δ?

ε_i, φ_i : string lengths for the $\mathfrak{sl}_2$-triples.

wt : the H-wts

A1. $\begin{cases} \underline{\mathrm{wt}(e_i \cdot x) = \mathrm{wt}(x) + \alpha_i} \\ \varepsilon_i(e_i \cdot x) = \varepsilon_i(x) - 1 \\ \varphi_i(e_i \cdot x) = \varphi_i(x) + 1 \end{cases}$

: $h\, e_i x = e_i h x + [h, e_i]\, x$

$= \mathrm{wt}(h)\, e_i x + \alpha_i(h)\, e_i x$

$= [\mathrm{wt}(h) + \alpha_i(h)](e_i x)$

: root string defs

$e_i(x) = y \iff f_i(y) = x$: wt spaces come in parts of dim 1; maybe there's more subtlety?

A2. $\varphi_i(x) = \langle \mathrm{wt}(x), \alpha_i^\vee \rangle + \varepsilon_i(x)$: For the sl_2-triple for $i \in I$,

$[h_i, x] = \mathrm{wt}(x)(h_i)(x)$

$= \langle \mathrm{wt}(x), \alpha_i^\vee \rangle (x)$.

Example 2.19. A_r

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \cdots \xrightarrow{r} \boxed{r+1}$$

(Convention: only draw edges $\boxed{x} \xrightarrow{i} \boxed{f_i(x)}$)

$\mathfrak{sl}_{r+1}$ action on $\mathbb{C}^{r+1}$.

Once we require seminormality, ε_i, φ_i can be inferred.

Example 2.22:

$$\boxed{1} \xrightarrow{1} \boxed{2} \longrightarrow \cdots \boxed{r} \xrightarrow{r} \boxed{0} \xrightarrow{r} \boxed{\bar{r}} \longrightarrow \cdots \longrightarrow \boxed{\bar{1}}.$$

$\mathfrak{so}(2l+1)$-action on $\mathbb{C}^{2l+1}$. $r = l$ for our earlier B_l computation and $\bar{i} = l + i$.

Similarly Examples 2.23 and 2.24 come from the standard modules of C_r and D_r.