Notes for Tues. Jan 28

I. Quick Recap

- A root system Φ is a collection of vectors in $\mathbb{R}^n$ that play nicely with associated reflections.

- From the definition of the reflection map, we get coroots, $\alpha^\vee$.

- From the root system, we can build a lattice on span Φ and call the elements in it weights.

- We can define fundamental weights that "pick out" the roots:

$$\langle \overline{\omega}_i, \alpha_j^\vee \rangle = \delta_{ij}$$

fundamental weight defined by this

$$\alpha_j^\vee = \frac{2\alpha_j}{\langle \alpha_j, \alpha_j \rangle} = \frac{2\alpha_j}{\alpha_j \cdot \alpha_j} = \frac{2}{|\alpha_j|^2}\alpha_j$$

II Discussion on Kashiwara crystals

Definition: Let Φ be a root system indexed by I, and let Λ be the weight lattice.

A <u>crystal</u> of type Φ is a nonempty set $\mathcal{B}$ (usually $\mathcal{O}_{\not\ni}$) together with maps

$$(i \in I) \quad \begin{array}{ll} e_i, f_i : \mathcal{B} \longrightarrow \mathcal{B} \sqcup \{0\} & \leftarrow \text{crystal operators} \\ \varepsilon_i, \varphi_i : \mathcal{B} \longrightarrow \mathbb{Z} \sqcup \{-\infty\} & \leftarrow \text{string lengths} \\ \mathrm{wt} : \mathcal{B} \longrightarrow \Lambda & \leftarrow \text{weight map} \end{array}$$

that satisfy the following:

- For $x, y \in \mathcal{B}$, $e_i(x) = y \iff f_i(y) = x$

  extra pieces: $\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i$, $\varepsilon_i(y) = \varepsilon_i(x) - 1$, $\varphi_i(y) = \varphi_i(x) + 1$

- $\varphi_i(x) = \langle \mathrm{wt}(x), \alpha_i^\vee \rangle + \varepsilon_i(x)$

  extra pieces: $\varphi_i(x) = -\infty \Rightarrow \varepsilon_i(x) = -\infty$ and we require $e_i(x) = f_i(x) = 0$

If $\mathcal{B}$ is a crystal, we associate a directed graph with labeled edges:

vertices: elements of $\mathcal{B}$

edges: $x \xrightarrow{i} y$ if $f_i(x) = y$

This description makes me think of f and e as operators that move "with" or "against" the direction of the graph but maybe that's backwards.

III Questions/Discussion

$\boxed{Q0}$ Open floor

Q1 There are a bunch of examples in the text. Are there any in particular we should discuss?

Ex 2.19

Q2 There are a lot of descriptors for crystals. What intuition can we gather for them?

Q3 Tensor products seem to matter in Ch 3. Do any particular exercises look better or worse for gaining familiarity?

Proof of 2.29 – HW for Friday

My answer to Q1 :

- Either today or Friday, do the following
  - ☆ Compute the crystal operators in Ex 2.19 (type A)
  - Discuss Ex 2.25 (row crystals) + Ex 2.26 (column crystals) at length as a preview for ch 3.
- What the heck makes a crystal "standard"?

My answer to Q3: 2.1 - boring but important?
2.2 - 50/50 on if it'll be good
2.4 and or 2.8

2.10 (unrelated to tensors but the first half seems interesting)