Chapter 12: the $\mathcal{B}_\infty$ crystal

Goal: Crystalize the Verma Modules $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\lambda)$ which are universal $U(\mathfrak{g})$-modules w/ highest weight $\lambda \in \Lambda$

Preliminaries

- The elementary Crystal $\mathcal{B}_i$

for α_i simple,

$$\mathcal{B}_i = \cdots \underset{u_i(-2)}{\bullet} \xleftarrow{i} \underset{u_i(-1)}{\bullet} \xleftarrow{i} \underset{u_i(0)}{\bullet} \xleftarrow{i} \underset{u_i(1)}{\bullet} \xleftarrow{i} \underset{u_i(2)}{\bullet} \cdots$$

$$\mathrm{wt}(u_i(n)) = n \cdot \alpha_i$$

$$\varphi_j(u_i(n)) = \begin{cases} n & \text{if } i=j \\ -\infty & \text{o/w} \end{cases}$$

$$\varepsilon_j(u_i(n)) = \begin{cases} -n & \text{if } i=j \\ -\infty & \text{o/w} \end{cases}$$

- The Crystal $\mathcal{T}_\lambda$ $(\lambda \in \Lambda)$ [Example 2.28]

$$\mathcal{T}_\lambda = \underset{t_\lambda}{\bullet}$$

No arrows! all e_i, f_i act as 0.

$$\mathrm{wt}(t_\lambda) = \lambda$$
$$\varphi_i(t_\lambda) = \varepsilon_i(t_\lambda) = -\infty$$

Levi Branching $\mathcal{B}_i$:

$$\mathrm{Branch}_{\{i\}} \mathcal{B}_i \cong \text{infinite } \mathfrak{sl}_2\text{-crystal}$$

$$\mathrm{Branch}_{\{j\}} \mathcal{B}_i = \bigoplus_{\mathbb{Z}} \mathcal{T}_0$$

Perhaps incorrect weight. Unclear from text

Tensor Products of elementary Crystals

<u>Fact:</u> $T_\lambda \otimes \mathcal{C}$ has the same Crystal graph, ε_i's as $\mathcal{C}$ [for any Crystal $\mathcal{C}$]

$x \in \mathcal{C}$, $\varphi_i(t_\lambda \otimes x) = \max(\underbrace{\varphi_i(t_\lambda)}_{=-\infty}, \varphi_i(x) + \underbrace{\langle wt(t_\lambda), \alpha_i^\vee \rangle}_{})$

if ↗ is nonzero, then $\varphi_i(t_\lambda \otimes x) \neq \varphi_i(x)$

$\varepsilon_i(t_\lambda \otimes x) = \max(\varepsilon_i(x), \underbrace{\varepsilon_i(t_\lambda)}_{=-\infty} + \cdots)$

<u>Prop</u> 12.2: $\mathcal{B}_i \otimes \mathcal{B}_i \cong \bigoplus_{k \in \mathbb{Z}} T_{k\alpha_i} \otimes \mathcal{B}_i$

<u>Prop</u> 12.3:
- $\mathcal{B}_i \otimes \mathcal{B}_j \cong \mathcal{B}_j \otimes \mathcal{B}_i$ "easy isom." if α_i, α_j orthogonal
- $\mathcal{B}_i \otimes \mathcal{B}_j \otimes \mathcal{B}_i \cong \mathcal{B}_j \otimes \mathcal{B}_i \otimes \mathcal{B}_j$ "hard isom." if $\langle \alpha_i, \alpha_j^\vee \rangle = -1$

Both isoms fix $u_i(0) \otimes \cdots \otimes u_i(0)$

<u>In Summary:</u> $\mathcal{B}_i$'s under tensor act a lot like the $s_i \in W$.
in particular, "reduced" products are all isom.

<u>Thm 12.6</u> if $w_0 = s_{i_N} \cdots s_{i_1}$ is the longest element in W,

$$\mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N}$$

is weakly Stembridge

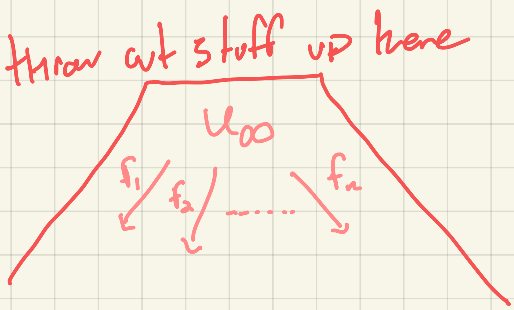
Proof: Levi Branch, tensor tricks

Def: The $\underline{\mathcal{B}_\infty \text{ Crystal}}$ is

$$\left\{ x \in \mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N} \;\middle|\; x \prec \underbrace{u_{i_1}(0) \otimes \cdots \otimes u_{i_N}(0)}_{\text{"}u_\infty\text{"}} \right\}$$

w/ $e_i(x) = 0$ if $\varepsilon_i(x) = 0$, all other e_i, f_i, φ_i, ε_i, wt as in the tensor prod.

Idea:



Thm 12.8: $\mathcal{B}_\infty$ is weak stembridge, upper seminormal

Thm 12.9 If $\mathcal{C}$ is conn. weak Stembridge Crystal

w/ highest weight elt. u_λ, $\mathrm{wt}(u_\lambda) = \lambda$,

$\exists! \; \mathcal{C} \hookrightarrow \mathcal{T}_\lambda \otimes \mathcal{B}_\infty$

$u_\lambda \longmapsto t_\lambda \otimes u_\infty$

Example: $A_2\,(GL(3))$

$w_0 = s_1 s_2 s_1$

$\mathcal{B}_\infty \subseteq \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1$

denote $u_1(a) \otimes u_2(b) \otimes u_1(c) = u(-a,-b,-c)$

So $u_\infty = (0,0,0)$

Using tensor formulas on Page 21:

$$f_2(a,b,c) = (a,b+1,c)$$

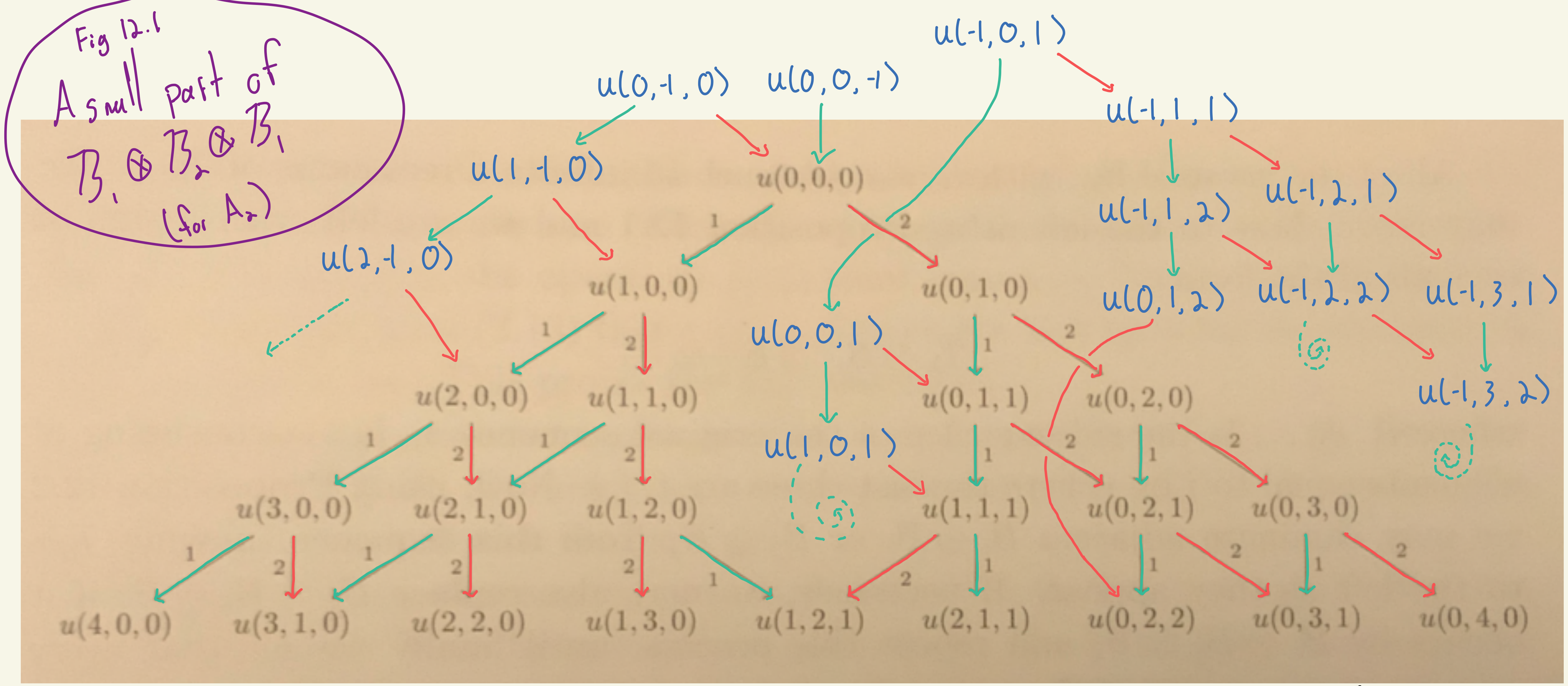
$$f_1(a,b,c) = \begin{cases} (a+1,b,c) & \text{if } b \le c+a \\ (a,b,c+1) & \text{if } b > c+a \end{cases}$$

Next time:

- Exercise 12.1 → Hunter

Fig 12.1

A small part of $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1$ (for A_2)



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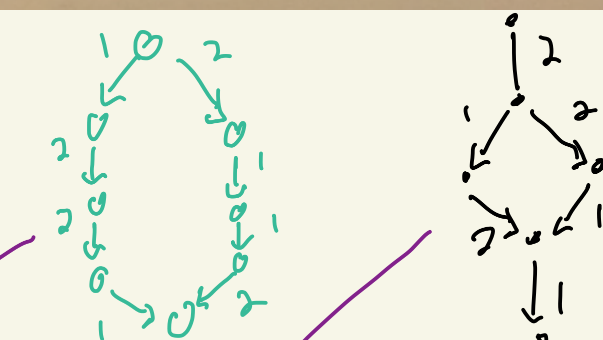
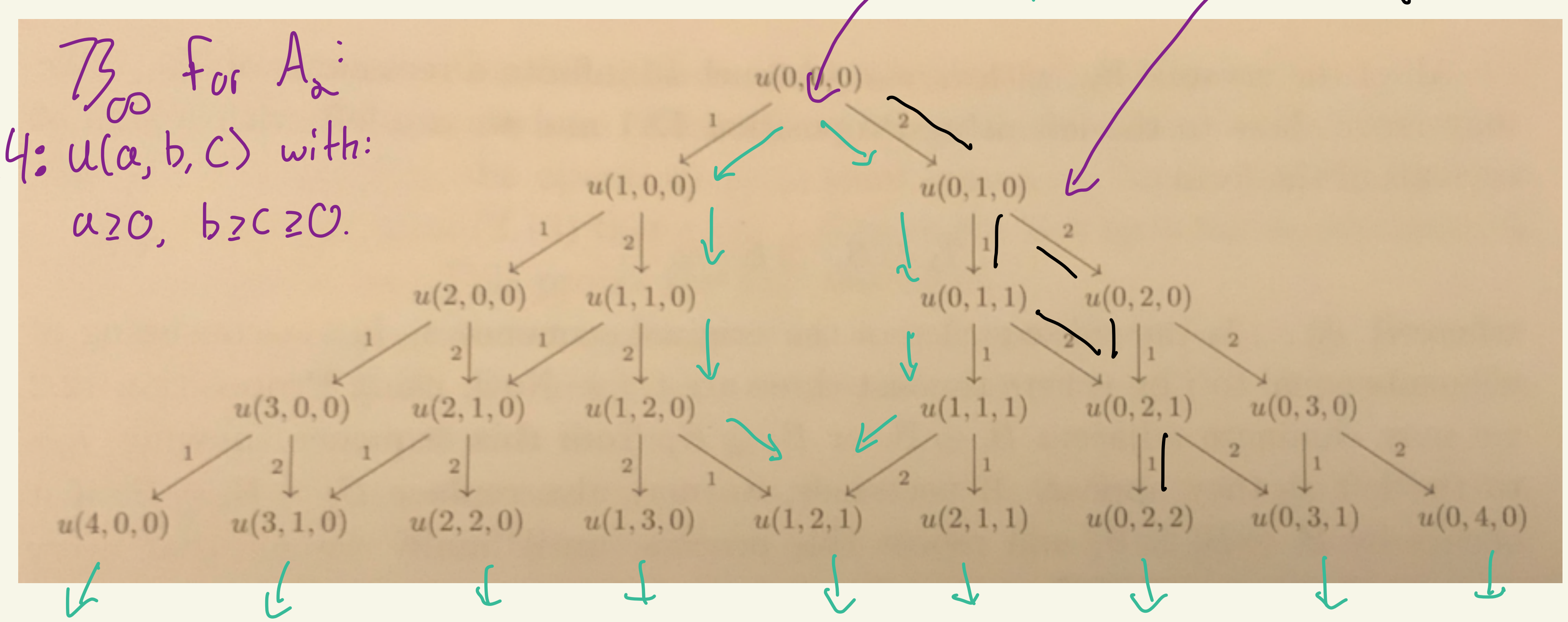
Excercise / Apply Theorem 12.9: $\mathcal{B}_{(2,1)}$



$\mathcal{B}_\infty$ for A_2:

Prop 12.4: $u(a,b,c)$ with: $a \ge 0,\ b \ge c \ge 0.$



Given $GL(n)$ tableaux T, T' of shape λ, λ' define an equivalence relation $T \equiv T'$ if for all $i \in [n]$, the i^{th} rows R_i, R_i' differ only by a string of i's at the beginning of the row.

We say a tableaux is *large* if the # of i's in R_i is strictly greater than the length $|R_{i+1}|$.

(i) Show if $T \equiv T'$ are large tableaux then e_i and f_i preserve $\equiv$, then give a crystal structure on equivalence classes of large tableaux such that a Yamanouchi tableaux has weight zero. (Verify axioms!)

(ii) Prove this crystal is isomorphic to $\mathcal{B}_\infty$. (See book hint).

(i) How does the signature rule work on large tableaux?

$$\begin{array}{|cccccccc|} \hline 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 4 & & & & \\ 3 & & & & & & & \\ \hline \end{array}$$ is large

$\shortparallel$

$$f_1 \left(\begin{array}{ccccccccccccc} 3 & 2 & 2 & 3 & 4 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ & (+ & (+ & & & -) & -) & - & - & - & + & + & \end{array} \right) = 3\ 2\ 2\ 3\ 4\ 1\ 1\ 1\ 1\ 2\ 2\ 3$$

$$= \begin{array}{|ccccccc|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 4 & & & \\ 3 & & & & & & \\ \hline \end{array} \leftarrow \underline{\text{Not}} \text{ large.}$$

$$e_1 \left(\begin{array}{ccccccccccccc} 3 & 2 & 2 & 3 & 4 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ & (+ & (+ & & & -) & -) & - & - & - & + & + & \end{array} \right)$$

$3\ 2\ 2\ 3\ 4\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 3$

- For large tableaux, the condition that there be more i's in R_i than entries in R_{i+1} combines w/ strict increasing columns so that f_i acts on either the rightmost $\boxed{i \text{ in } R_i}$ or acts on an unbracketed i in the "tail" part of a row above.
  can produce a non-large tableaux
- Can choose a representative $\tilde{T} \equiv T$ that $f_i \tilde{T}$ is large: insert a column $\begin{array}{|c|} \hline 1 \\ 2 \\ \vdots \\ i \\ \hline \end{array}$ to T.

  if $T_1 \equiv T_2$ are large and $f_i T_1, f_i T_2$ are both large, then $f_i T_1 \equiv f_i T_2$

  Cases ① f_i acts on the rightmost i in $R_i(T_1) \Rightarrow$ no unpaired i in rows $R_j(T_1)$ for $j<i \Rightarrow$ same situation in T_2 since $T_1 \equiv T_2$.

  ② f_i acts on unpaired i in $R_j(T_1)$ for $j<i$ $\Rightarrow$ the same i is unpaired in $R_j(T_2)$.

- e_i acts as zero (our crystal will be upper seminormal)

  or $e_i T$ is large.

  ↳ e_i acts on the leftmost unpaired $(i+1)$, which can only belong to the "tail" part of T since largeness pairs all $(i+1)$'s in R_{i+1} w/ i in R_i. } clear that this preserves equivalence.

  ↳ this is our main evidence that Yamanouchi Tableaux are still of highest weight.

- Define $\widetilde{wt}(T) = wt(T) - wt(\lambda)$ where wt is the usual weight on tableaux and λ is Yamanouchi in the shape of T.

  - Well-defined on equivalence classes since $|(\widetilde{wt}(T))_i| = \#$ of i's in the "tail" part of T.

- Define $\varepsilon_i([T]) = \varepsilon_i(T)$ (so we will have upper seminormality)

  and $\varphi_i([T]) = \varepsilon_i([T]) + \langle wt([T]), \alpha_i^\vee \rangle$.

- Can verify that $f_i([T]) = [T'] \iff e_i([T']) = [T]$

  Ex | If $f_i([T]) = [T']$ then

  $$e_i([T']) = e_i(f_i[T]) = e_i([f_i(T)]) = [e_i \circ f_i(T)] = [T]$$

  since e_i, f_i preserve equivalence classes.

(2) Isomorphism with $\mathcal{B}_\infty$:

- Hint in book: use $w_0 = (s_{n-1})(s_{n-2}s_{n-1}) \cdots (s_1 s_2 \cdots s_{n-1})$

  so that $\mathcal{B}_\infty$ embeds into $\mathcal{B}(n-1) \otimes \cdots \otimes \mathcal{B}(1)$

  where $\mathcal{B}(i) = \mathcal{B}_i \otimes \mathcal{B}_{i+1} \otimes \cdots \otimes \mathcal{B}_{n-1}$ and relate eq. classes of i^{th} rows to $\mathcal{B}(i)$.