Ch 10. Crystals for Stanley Symm functions

— how establishing an $A_{\ell-1}$ crystal structure on a set (decreasing factorizations) pays off.

Main result: Combinatorial interpretation of → Schur positivity. Interpretation of a_λ

$$(*) \quad \underset{\substack{\downarrow \\ \text{S.S.F.}}}{f(x)} = \sum_{\lambda} \underset{\substack{\downarrow \\ \in \mathbb{N}}}{\underline{a_\lambda}}\, S_\lambda(x)$$

Idea: $f(x)$ will be the char. of an $\underline{A_{\ell-1} \text{ crystal}}$

$\downarrow$

iso to B_λ, whose char

is $S_\lambda(x) = \sum_{T \in B_\lambda} t^{wt(T)}$

So $(*)$ will follow from a char decategorification of a crystal isomorphism

$$\mathcal{B}_{f(x)} \cong \bigoplus_{\lambda} B_\lambda^{\oplus a_\lambda}$$

Stanley symm. functions.

decreasing factorization of an elt $w \in S_n$: reduced word

$$w = w^k w^{k-1} \cdots w^1 \quad \text{w/} \quad \ell(w) = \ell(w^k) + \cdots + \ell(w^1)$$

each w^i is decreasing ($s_5 s_3 s_1$ ✓ $s_3 s_2 s_4$ ×), possibly empty.

Notation: W_w (will be our crystal) $:= \{$decreasing factorizations of $w\}$.

often infinite since w^i can be empty.

$W_w^{(\ell)} = \{$decreasing factorizations w/ ℓ parts$\}$

finite

e.g. $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2 \in S_3 = A_2$, $\ell = 3$ (Eg. 10.1)

$$\rightsquigarrow W^3_{w_0} = \left\{ \begin{array}{l} (1)\,(2)\,(1),\ (2)\,(1),\,(2) \\ (\ )\,(1)\,(21),\ (1)\,(\ )\,(21),\ (1)(2)(1), \\ (\ )\,(21)\,(2),\ (21)\,(\ )\,(2),\ (21)(2)(\ ) \end{array} \right.$$

→ more generally, if $\ell = \ell(w)$, { a dec. fact w/ ℓ parts $\equiv$ a red. word }

but not $(12)(\ )(1)$, $(12)(1)(\ )$, etc

Def. S.S.F.
$$F_w(x) = \sum_{w^k \cdots w^1 \in W_w} x_1^{\ell(w^1)} \cdots x_k^{\ell(w^k)}$$
inf. many variables

↙ restrict to ℓ variables

note: not $x_1^{\ell(w^\ell)}$

$$F^\ell_w(x) = \sum_{w^\ell \cdots w^1 \in W^\ell_w} x_1^{\ell(w^1)} \cdots x_\ell^{\ell(w^\ell)}$$

The trick: specialize a symm func. $f(x) = \sum_\lambda a_\lambda S_\lambda(x)$ by setting $x_i = 0$ $\forall\, i > \ell$ for fixed ℓ will give

$$f(x_1, \dots, x_\ell) = \sum_{\mathrm{len}(\lambda) \le \ell} a_\lambda S_\lambda(x)$$

since $S_\lambda \equiv 0$ if $\mathrm{len}(\lambda) > \ell$

eg. In the above example
$$F^3_{w_0}(x_1, x_2, x_3) = 2x_1x_2x_3 + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$$

General problem for me: understand the standard trick to deduce properties of symm. function by studying their restrictions to symm. polynomials.

Nonobvious Facts.

Thm 10.2 (Stanley 1984) (1) $F_w(x)$ is a symm.

(However, this would follow easily from the crystal structure on W_w ($\mathrm{char}\, W_w = F_w(x)$) by Prop 2.37)

(2) " $F_w(x)$ is concentrated on an interval $[\lambda(w), \mu(w)]$ for some λ, μ, and more ... "

Thm 10.3 (Edelman – Greene 1987) $a_{w,\lambda} \in \mathbb{Z}_{\geq 0}$ (Schur positivity)

More. (Haiman 1992, Reiner-Shimozono 1998) Combinatorial interpretation of $a_{w,\lambda}$

Crystal Interpretation — Morse and Schilling (2016)

So what is the crystal structure on W_w^ℓ? $\mathrm{wt}(w^\ell w^{\ell-1} \cdots w^1) = (\ell(w^1), \ldots, \ell(w^\ell))$

Need: define the Kashiwara operators on dec factorizations. check axioms

will get: "local" Stembridge crystals $\searrow$ the axioms are mostly easy to check.

e_i / f_i only affect w^{i+1}, w^i

Key ingredient: pairing.

Def. Take input $w^\ell w^{\ell-1} \cdots w^1$. Eg. 10.5 eg. $w = \underbrace{(s_3 s_2)}_{w^3} \underbrace{(s_3 s_1)}_{w^2} \underbrace{(s_2)}_{w^1} \in S_4$

To define f_i / e_i, consider only w^i and w^{i+1}. eg. $i=2$.

· 'Pairing' attempts to associates an $a \in w^i$ to each $b \in w^{i+1}$, taking the b's in decreasing order. a should be smallest remaining entry $> b$ in w^i.

first, take $b = s_3$, no a can be selected $(b,a) = (s_3, \phi)$

then, take $b = s_2$, $a = s_3$ $(b,a) = (s_2, s_3)$

- Consider $L_i = \{ b \in w^{i+1} \mid b \text{ is unpaired in the } (i+1)\text{-}i \text{ pairing} \}$

$R_i = \{ a \in w^i \mid a \text{ is unpaired in the } (i+1)\text{-}i \text{ pairing} \}$

e.g. $L_i = \{ s_3 \}$ $R_i = \{ s_1 \}$.

$w^{i+1} = s_3 s_2$

- def. of e_i : $\begin{cases} \underline{\text{if}} \quad L_i = \phi, \ e_i \text{ acts as } 0. \\ \underline{\text{else}}, \ \text{take } b = \min(L_i), \ t = \min\{ j \geq 0 : b-j-1 \notin w^{i+1} \} \end{cases}$

then take b away from w^{i+1}, add $b-t$ to w^i (inserting it in the correct place)

e.g. $b = 3$, $b-1=2$, so $t=1$, so

$$e_2\big((s_3 s_2)(s_3 s_1)(s_2) \big) = (s_2)(s_3 s_2 s_1)(s_2)$$

braid

Facts.

Lemma 10.4. "the operators are well-defined".

- Cont / supp of w^{i+1} contains $b, b-1, \ldots, b-t$ but not $b-t-1$.
- . . . w^{i} . . . . $b, b-1, \ldots, b-t+1$ but not $b-t$ or $b+1$.

follows from minimality ?

- $w^{i+1} w^{i} = \widetilde{w^{i+1}} \widetilde{w^{i}}$ (so we are still in W^{ℓ}_{w})

Thm 10.6 $B(w) := W^{\ell}_{w}$ is Stembridge crystal of type $A_{\ell-1}$.

- should be relatively easy to check given the 'local' actions.
- leads to h.w. theory, easy to determine iso types by h.w.

10.3 Applications

dominant wts. $\Leftrightarrow$ partitions

10.3. Applications

Notation: $W^{\ell}_{w,\lambda} = \{ \text{elts in } W^{\ell}_{w} \text{ of wt } \lambda \}$.

Corollary 10.8. If (A) $F_w(x) = \sum_{\lambda} a_{w,\lambda} S_\lambda(x)$, → symm func. inf. many variables

then (B) $a_{w,\lambda} = \left| \{ \underset{\substack{\| \\ x}}{w^{\ell} \cdots w^{1}} \in W^{\ell}_{w,\lambda} \mid \underset{\text{(for this } A_{\ell-1} \text{ crystal)}}{x \text{ is } h.w} \} \right|$ → symm poly. variables $x_1, \dots, x_\ell$

should follow immediately from wt considerations, by choosing ℓ sufficiently large

(Q: standard restriction trick ✓) char $\left(\bigoplus B_\lambda^{\oplus a_{w,\lambda}}\right)$

Specializing to $x_i = 0$ $\forall i > \ell$, (A) gives //

(C) $F^{\ell}_{w}(x) = \sum_{\substack{\lambda \\ \text{len}(\lambda) \leq \ell}} a_{w,\lambda} S_\lambda(x)$ $\left(\text{because } S_\lambda(x) = 0 \text{ when } \text{len}(\lambda) > \ell\right)$

$\sum_{d \in W^{\ell}_{w}} x^{\text{wt}(d)} \underset{\text{def}}{:=}$ $\overset{\|}{\text{char } W^{\ell}_{w}}$ (because each dec fact $w^{\ell} \cdots w^{1}$ contributes $x_1^{\ell(w^1)} \cdots x_\ell^{\ell(w^\ell)}$ to both)

(c) $$F_w^{\ell}(x) = \sum_{\substack{\lambda \\ \mathrm{len}(\lambda)\le \ell}} a_{w,\lambda} S_\lambda(x)$$

Now in (c), $$F_w^{\ell}(x) \overset{\text{def}}{=} \sum_{w^{\ell}\cdots w^{1}\in W_w^{\ell}} x_1^{\ell(w_1)}\cdots x_\ell^{\ell(w_\ell)}$$

$$= \sum_{w^{\ell}\cdots w^{1}\in W_w^{\ell} =: B(w)} \mathbf{x}^{wt(w^1\cdots w^{\ell})}$$

$$\overset{\text{def}}{=} \mathrm{char}(B(w))$$

$$= \sum_{\substack{\lambda \\ \mathrm{len}(\lambda)\le\ell}} c_{w,\lambda} S_\lambda(x) \quad \text{if} \quad B(w) \cong \bigoplus_{\substack{\lambda \\ \mathrm{len}(\lambda)\le\ell}} B_\lambda^{\oplus c_{w,\lambda}}$$

So, since Schur poly form a basis of symm poly, it follows that $a_{w,\lambda} = c_{w,\lambda}$.

ie, $$a_{w,\lambda} = \text{multiplicity of } B\lambda \text{ in } B(w)$$
$$= \#\text{ h.w elts w/ wt } \lambda \text{ in } B(w) \rightarrow (B)$$

$\square$

Crystal Isomorphism $\mathcal{B}(w) \simeq \bigoplus_{\lambda} B_{\lambda}^{\oplus a_{w,\lambda}}$

- Takes connected components of the crystal on decreasing factorizations to an isomorphic crystal of tableaux.

① Turn a decreasing factorization $w = v^{\ell} v^{\ell-1} \cdots v^{1}$ to an increasing factorization $\bar{v}^{1} \bar{v}^{2} \cdots \bar{v}^{\ell}$ by reversal.

② EG-insert the factors: $\emptyset \leftarrow \bar{v}^{1} \leftsquigarrow \bar{v}^{2} \leftsquigarrow \cdots \leftsquigarrow \bar{v}^{\ell}$, with the insertion tableau as usual, but for Q, record each new box created in the $\bar{v}^{i}$ insertion as an "i".

- Yields a map $\Phi_{EG} : v^{\ell} \cdots v^{1} \mapsto (P, Q)$

$$\mathcal{B}(w) \cong \bigoplus_{\lambda} B_{\lambda}^{\oplus a_{w,\lambda}}$$

- $\Phi_{EG}^{rec} : v^{\ell} \cdots v^{1} \mapsto Q$ is the crystal isomorphism.

Example: Let $w = s_1 s_2 s_3 s_2$, factor as $v^3 v^2 v^1 = (1)(2)(32)$.

Get ascending factorization $\bar{v}^1 \bar{v}^2 \bar{v}^3 = (23)(2)(1)$

$\emptyset \leftarrow \bar{v}^1$: $\boxed{2} \leftsquigarrow 3 \to P = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}$ $\quad Q = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$

$\emptyset \leftarrow \bar{v}^1 \leftarrow \bar{v}^2$: $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array} \leftsquigarrow 2 \to P = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$ $\quad Q = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}$

$\emptyset \leftarrow \bar{v}^1 \leftarrow \bar{v}^2 \leftarrow \bar{v}^3$: $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \leftsquigarrow 1 \to P = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$ $\quad Q = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$

Another factorization: $w = s_1 s_2 s_3 s_2 = s_1 s_3 s_2 s_3 = s_3 s_1 s_2 s_3$, $\quad v^3 v^2 v^1 = (31)(2)(3)$

$\bar{v}^1 \bar{v}^2 \bar{v}^3 = (3)(2)(13)$

$\emptyset \leftarrow \bar{v}^1$: $P = \begin{array}{|c|}\hline 3 \\ \hline \end{array}$ $\quad Q = \begin{array}{|c|}\hline 1 \\ \hline \end{array}$

$\emptyset \leftarrow \bar{v}^1 \leftarrow \bar{v}^2$: $3 \leftsquigarrow 2$ $\quad P = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}$ $\quad Q = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$

$\emptyset \leftarrow \bar{v}^1 \leftarrow \bar{v}^2 \leftarrow \bar{v}^3$: $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} \leftsquigarrow 1 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \leftsquigarrow 3 \to P = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$ $\quad Q = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$

$$wt(v^{\ell} \cdots v^{1}) = (\ell(v^1), \ell(v^2), \ldots, \ell(v^{\ell}))$$

Before drawing a subcrystal of $B(w)$, note how f_i acts on decreasing factorizations:

f_i acts on $w^{i+1}w^i$ by:

① Pair largest $b \in \mathrm{cont}(w^{i+1})$ with smallest $a \in \mathrm{cont}(w^i)$ s.t. $a > b$

② Repeat ① until no more pairs can be made.

③ Set $a =$ max unpaired letter in $\mathrm{cont}(w^i)$, $\boxed{s = \min\{j \geq 0 \mid a+j+1 \notin \mathrm{cont}(w^i)\}}$

If no unpaired $a \in \mathrm{cont}(w^i)$, $f_i = 0$.

④ Replace $w^{i+1}w^i$ with $\tilde{w}^{i+1}\tilde{w}^i$ where $\mathrm{cont}(\tilde{w}^{i+1}) = \mathrm{cont}(w^i) \cup \{a+s\}$ and $\mathrm{cont}(\tilde{w}^i) = \mathrm{cont}(w^i) \setminus \{a\}$ and $\tilde{w}^i, \tilde{w}^{i+1}$ are still decreasing

Example: $w = s_1 s_2 s_3 s_2$, $v^3 v^2 v^1 = (1)(2)(32)$

$f_1((1)(2)(32))$: $(2)(32)$ $a = 2$, $s = 1$

$\tilde{v}^2 = (32)$, $\tilde{v}^1 = (3)$

$f_1((1)(2)(32)) = (1)(32)(3)$

Crystal Graph: For the word $w = s_1 s_2 s_3 s_2$, the factorization $(1)(2)(32)$ has highest weight $(2,1,1)$ ← length of factors $(\ell(v^1), \ell(v^2), \ell(v^3))$

$$(1)(2)(32) \xrightarrow{1} (1)(32)(3) \xrightarrow{2} (31)(2)(3)$$

$$s_1 s_2 s_3 s_2 \xrightarrow{\text{braid}} s_1 s_3 s_2 s_3 \xrightarrow{\text{comm.}} s_3 s_1 s_2 s_3$$

Q: $$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \xrightarrow{1} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \xrightarrow{2} \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

Corollary of Crystal Isomorphism: Given $w \in S_n$ and λ, there is a bijection:

$$\left(\begin{array}{c}\text{highest weight factorizations} \\ v^\ell \cdots v^1 \in \mathcal{W}^\ell_{w,\lambda}\end{array}\right) \longleftrightarrow \boxed{\left(\begin{array}{c}\text{S.S. tableaux of shape } \lambda' \\ \text{whose column reading gives a} \\ \text{reduced word of } w\end{array}\right)}$$

The bijection is given by taking the conjugate of the insertion tableaux $P^{ins}_{EG}(v^\ell \cdots v^1) = P$.

Example/ $v^3 v^2 v^1 = (1)(2)(32)$, $\quad \mathcal{P}^{ins}_{EG}\big((1)(2)(32)\big) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} 3 & \\ \cline{1-1}\end{array}$

$s_1 s_2 s_3 s_2$
$\downarrow$
$s_1 s_3 s_2 s_3$
$\downarrow$
$s_3 s_1 s_2 s_3$

Conjugate $P' = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & & \\ \cline{1-1}\end{array}$, column reading is $\boxed{3123.}$

Note that the reading word given is **not** the word given to $\mathcal{P}^{ins}_{EG}(\ )$, and in fact $s_3 s_1 s_2 s_3$ is a **least** weight element in the same component of $\mathcal{B}(w)$.

$CK(w)$ = graph of reduced words, w/ edges between words related by one braid or Knuth relation

Fact/ Given $w \in S_n$, there is a one-to-one correspondence

$$\left(\begin{array}{c}\text{components of}\\ \text{Coxeter-Knuth graph}\\ CK(w)\end{array}\right) \longleftrightarrow \left(\begin{array}{c}\text{components of the}\\ \text{crystal } \mathcal{B}(w)\end{array}\right)$$

Example/ $v^3 v^2 v^1 = (2)(321)(2)$

$wt = (1,3,1)$

$\mathcal{P}^{ins}_{EG}\big((2)(321)(2)\big)$

$\emptyset \leftarrow (2) \leftarrow (123) \leftarrow (2)$

$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \longleftarrow 1$

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \longleftarrow 2$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array} \longleftarrow 3$

$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & & \\ \cline{1-1}\end{array} \longleftarrow 2$

$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array}$ $\qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array} \rightarrow (3\ 2\ 1\ 3\ 2)$

$(1,2,2)$