8.3

i) find two tableau of of shape $(2,2,1)/(1)$ whose r.w. are Yamanouchi

Deduce $B_{(2,2,1)/(1)} \cong B_{(2,2)} \oplus B_{(2,1,1)}$

- $\begin{array}{|c|c|} \hline & 1 \\ \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} \rightsquigarrow 2, 1, 2, 1$

  $\equiv_K 2, 2, 1, 1 \rightsquigarrow \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$

- $\begin{array}{|c|c|} \hline & 1 \\ \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \rightsquigarrow 3, 1, 2, 1$

  $\equiv_K 3, 2, 1, 1 \rightsquigarrow \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}$

$\rightsquigarrow B_{(2,2,1)/(1)} \cong B_{(2,2)} \oplus B_{(2,1,1)}$

by prop 8.10

- Why are these the only two?

$\begin{array}{cccc} 3 & & & \\ 2 & 1 & 2 & 1 \\ \_ & \_ & \_ & \_ \end{array}$

(first position) · 1, 2, 3 by Yam. · 1 isn't allowed due to tableau

(second position) · 1 or 3 by Yam · 1 by tableau

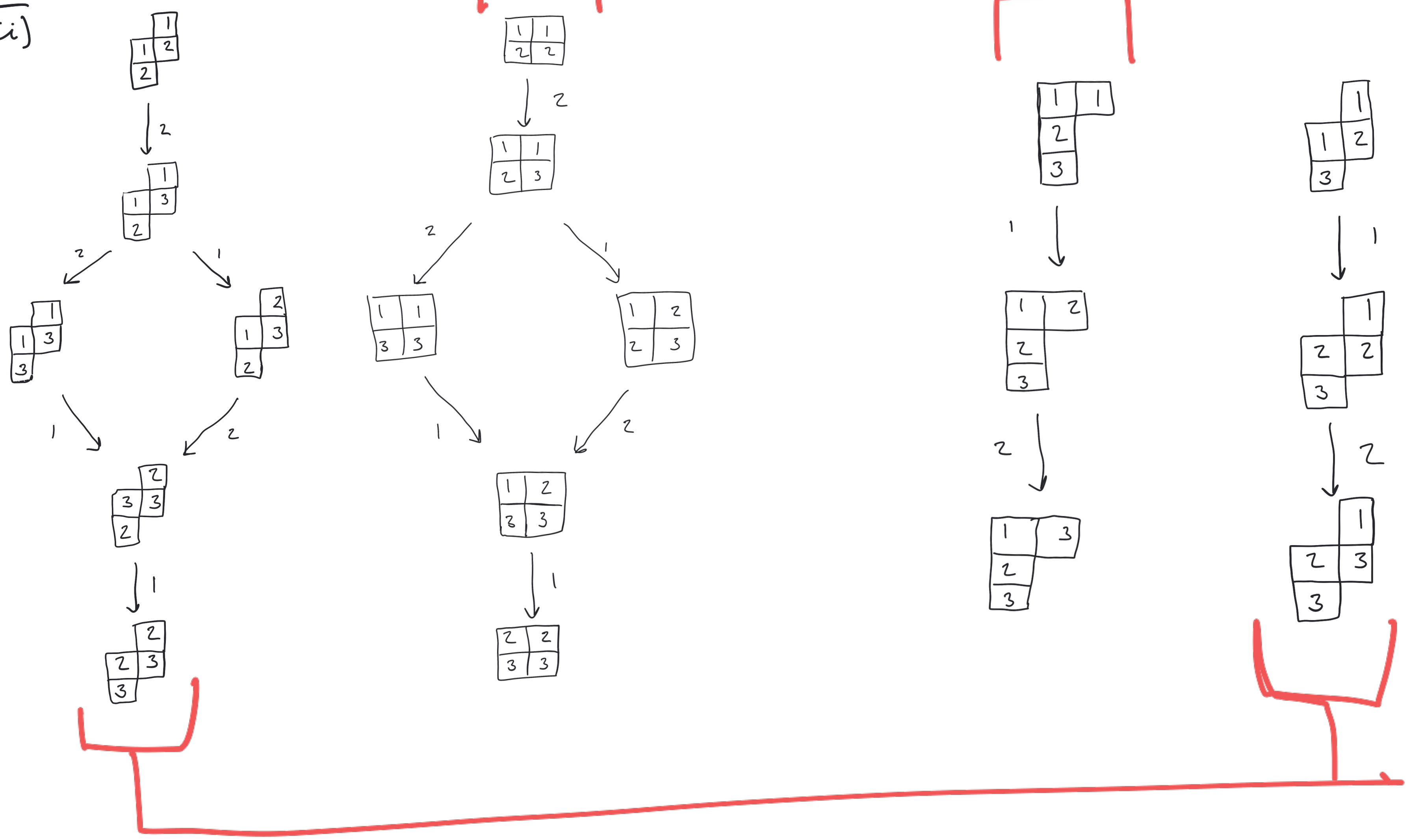
(third position) · 1 or 2 by Yam · 2 due to tableau restriction

(fourth position) has to be 1 by Yam.

B.3
ii)

$B_{(2,2)}$

$B_{(2,1,1)}$



$B_{(2,2,1)/(1)}$

$$\boxed{\begin{array}{l} \cdot\ 2121 \\ \cdot\ 2211 \\ \cdot\ 3211 \\ \cdot\ 3121 \end{array}}$$

· 1321

· 1111

· 2111

· 1211

· 1121

8.3

iii) for $n=3$ explain how these crystals can be embedded into $\mathbb{B}^{\otimes 4}$ for $(2,2)$ we have RR and CR, describe them both

???

8.4

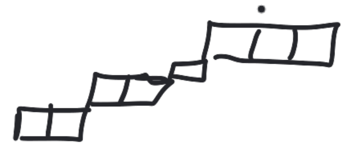
a)

· $c^{\lambda}_{\mu(K)} = \begin{cases} 1 & \lambda/\mu \text{ is a horizontal strip} \\ 0 & \text{o/w} \end{cases}$

# of ways to fill λ/μ so that the weight is the same as (K) and the row reading is Yam.

· $(K) = \boxed{1}\boxed{1}\boxed{\cdots}\boxed{1}$

· λ/μ



· theres only 1 way to put all 1's in here so

$c^{\lambda}_{\mu(K)} = 1$

· not a strip means there

$j > i \implies$ can't be all 1's so

$c^{\lambda}_{\mu(K)} = 0$

$wt(K) = 1^K$

$wt(1^K) = 1^1 2^1 \ldots K^1$

$(K, 0, 0, \ldots 0)$

$(1, 1, \ldots, 1)$

- only one way to put k
  strips in k boxes and
  we can def. do it
  for a vert. strip
- <u>NOT</u> a vert. strip

  - $\boxed{i \mid j}$ $\quad j \geq i$
  - if $j = i$, we have a
    duplicate so we can't
    have the same weight
  - if $j > i$, then the reading
    word has

    $\ldots \; i \; j \ldots$

    so either Not Yam or
    there is an i somewhere in

8.5 $\mu \vdash k, \quad \lambda \vdash (k+r) \qquad YD(\lambda) \supset YD(\mu)$

$\ell(\lambda) \leq n$

$T \in B_\mu$ such that $T' = T \leftarrow r$

$\leftarrow (r-1)$

$\vdots$

$\leftarrow 2 \leftarrow 1$

is the highest weight

T' of B_λ

$\Longleftrightarrow$ λ/μ is a vertical strip

Ex

$$T = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline 4 & 4 \\ \hline \end{array} \leftarrow 3 \leftarrow 2 \leftarrow 1 \qquad T' = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 4 & 4 \\ \cline{1-2} \end{array}$$

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 4 \\ \hline 4 & 5 \\ \hline \end{array} \leftarrow 3 \leftarrow 2 \leftarrow 1 \qquad \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

$\longrightarrow Y$

$I^X \longrightarrow I^Y/\mathrm{aut}$

$i \in I^X$, define γ_i:

arrow points to i:

$\gamma_i = 1$

arrow points away from i:

γ_i = size of arrow

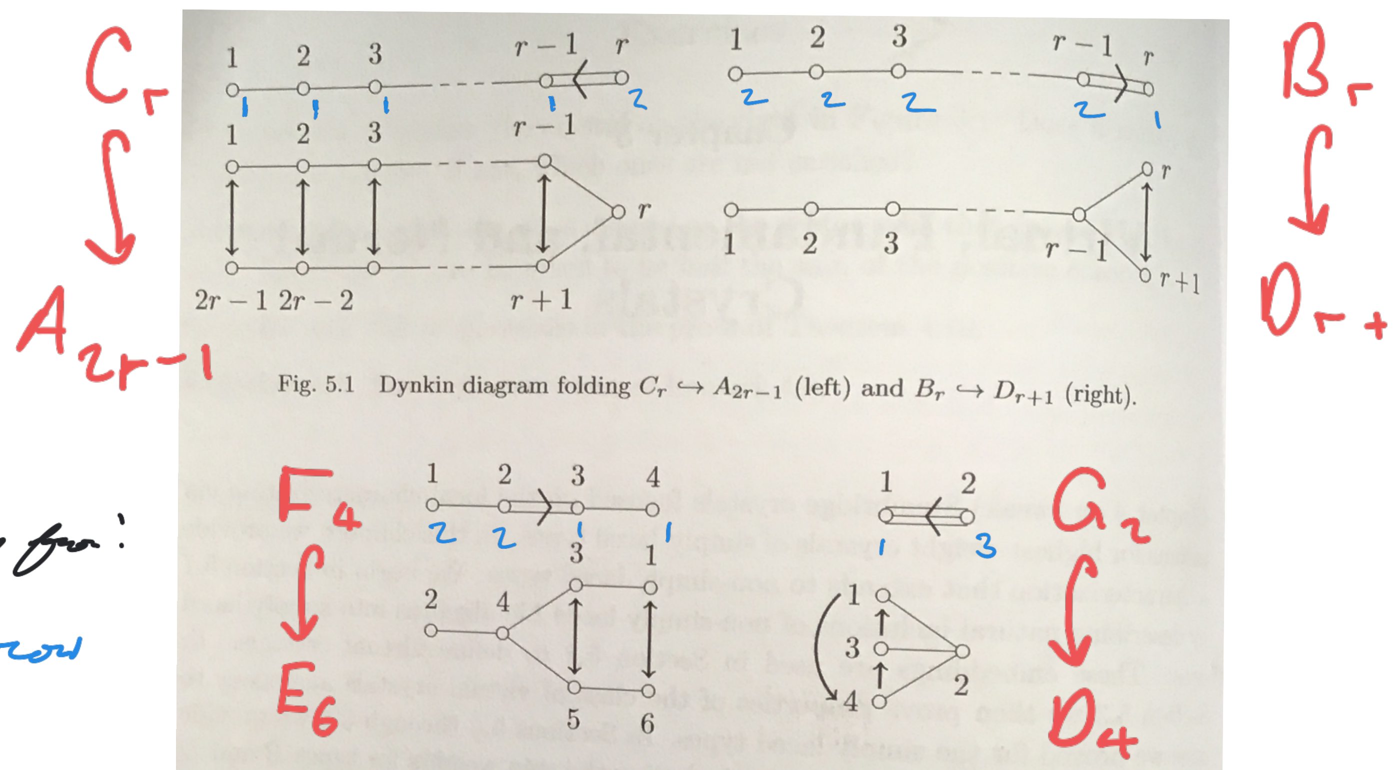


Fig. 5.1 Dynkin diagram folding $C_r \hookrightarrow A_{2r-1}$ (left) and $B_r \hookrightarrow D_{r+1}$ (right).

Fig. 5.2 Dynkin diagram folding $F_4 \hookrightarrow E_6$ (left) and $G_2 \hookrightarrow D_4$ (right).

weight lattice Λ_Y of Y to the weight lattice

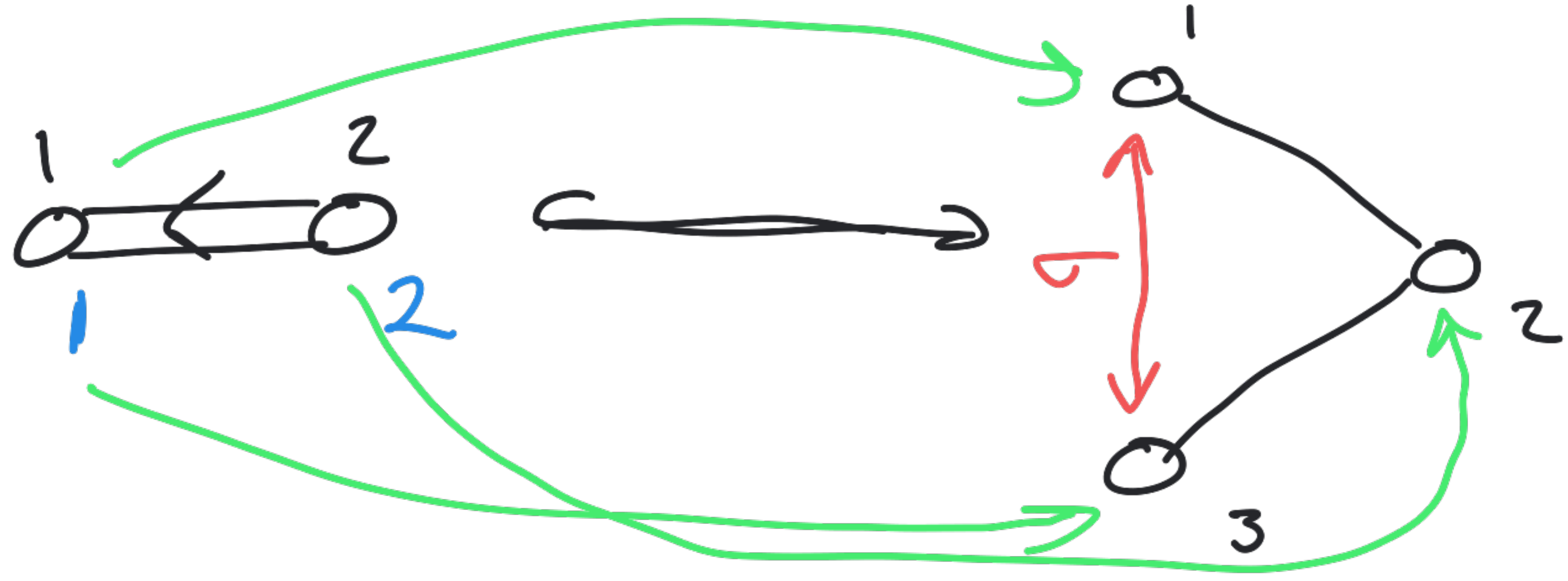
this gives

$$\Phi : \Lambda^X \longrightarrow \Lambda^Y$$

$$\varpi_i^X \longmapsto \gamma_i \sum_{j \in \sigma(i)} \varpi_j^Y$$

$$\alpha_i^X \longmapsto \gamma_i \sum_{j \in \sigma(i)} \alpha_j^Y$$

Ex



$$C_2 \longrightarrow A_3$$

tells us stuff about lengths of roots

$$\Phi(\bar{\omega}_1^{C}) = 1\,\bar{\omega}_1^{A} + 1\,\bar{\omega}_3^{A}$$

$$\Phi(\bar{\omega}_2^{C}) = 2\,\bar{\omega}_2^{A}$$

5.2

- assume Λ^X, Λ^Y are s.s. (spanned by fundamental weights)
- $X \hookrightarrow Y$ $\hat{V}$ a crystal of type Y,

  w/ operators $\hat{e}_i, \hat{f}_i$ for $i \in I$
- $\hat{V}$ the "ambient crystal
- $b \in \hat{V}$ has $\widehat{wt}(b)$, $\hat{\varphi}_i(b)$, $\hat{\varepsilon}_i(b)$
- for $i \in I^X$, we get virtual crystal operators:

$$e_i := \prod_{j \in \sigma(i)} \hat{e}_j^{\gamma_i} \qquad f_i := \prod_{j \in \sigma(i)} \hat{f}_j^{\gamma_i}$$

Virtual Crystal axiom $V \subseteq \hat{V}$

V1: $\hat{V}$ is stembridge

V2: $b \in V$, $i \in I^X$, then

- $\hat{\varepsilon}_j(b)$ is const. on $j \in \sigma(i)$
- $\hat{\varepsilon}_j(b)$ is a multiple of γ_i
- same for $\hat{\varphi}_j(b)$

V3: $V \cup \{0\}$ is closed under

$e_i, f_i \quad \forall i \in I^X$

- $\varepsilon_i(b) = \max\{k : e_i^k(b) \neq 0\}$
- $\varphi_i(b) = \max\{k : f_i^k(b) \neq 0\}$

this gives:

- $\varepsilon_i(b) = \frac{1}{\gamma_i} \hat{\varepsilon}_j(b)$
- $\varphi_i(b) = \frac{1}{\gamma_i} \hat{\varphi}_j(b)$

Ex 5.5

· $\hat{V} = B_{\bar\omega_1^A} \otimes B_{\bar\omega_3^A} = \square \times \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$ in A_3

(1,0,0) (1,1,1)

$\bar\omega_1 = e_1$

$\bar\omega_3 = e_1 + e_2 + e_3$

· $B_{\bar\omega_1^C}$ in C_2 is generated by $u_{\bar\omega_1^A} \otimes u_{\bar\omega_3^A}$ w/ operators $f_1 = \hat f_1 \hat f_3$, $f_2 = \hat f_2^{\,2}$

From Example 2.23

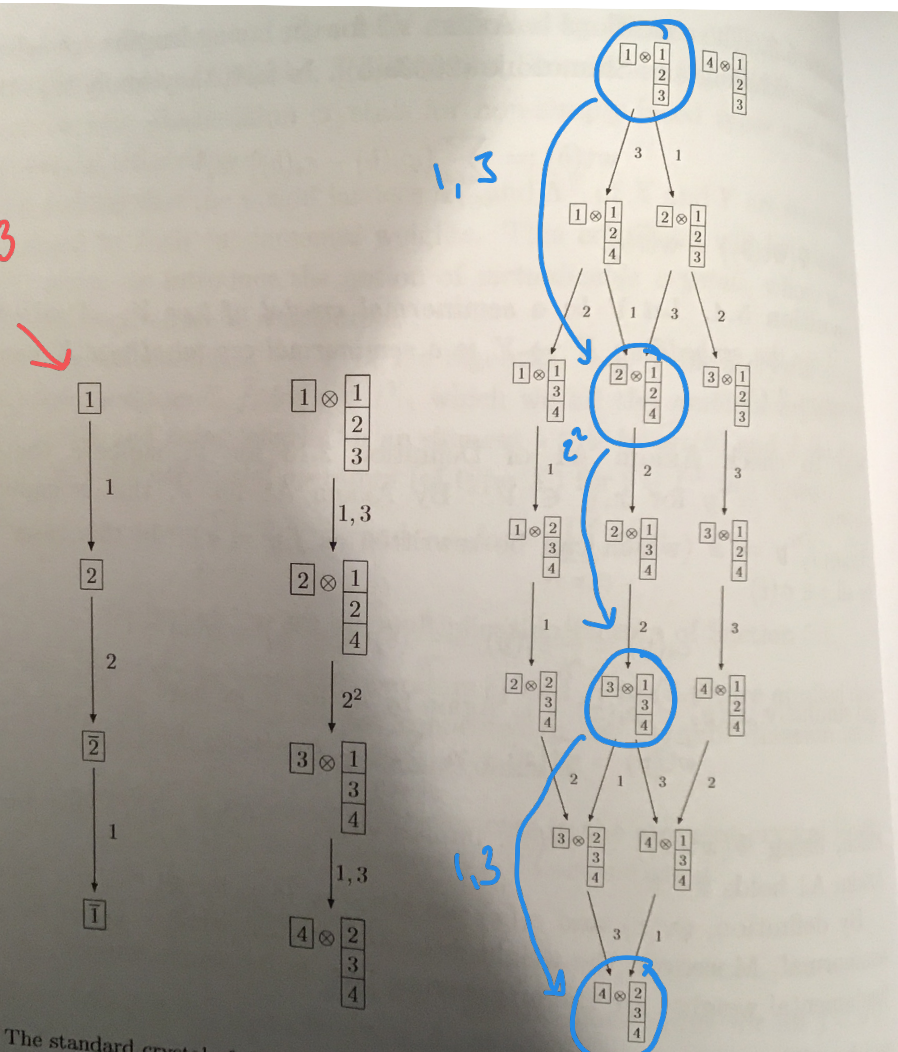


Fig. 5.3 The standard crystal of C_2 as a virtual crystal in A_3. Left: the C_2 crystal $\mathcal{B}_{\varpi_1}$. Right: the A_3 crystal $\mathcal{B}_{\varpi_1} \otimes \mathcal{B}_{\varpi_3}$, with two components. Middle: the virtual crystal.

Example 5.6. *Let us now consider the Lie alg*
$\hat{\mathcal{V}} = \mathcal{B}_{\varpi_1} \otimes \mathcal{B}_{\varpi_1}$ *of type* D_4. *Th*

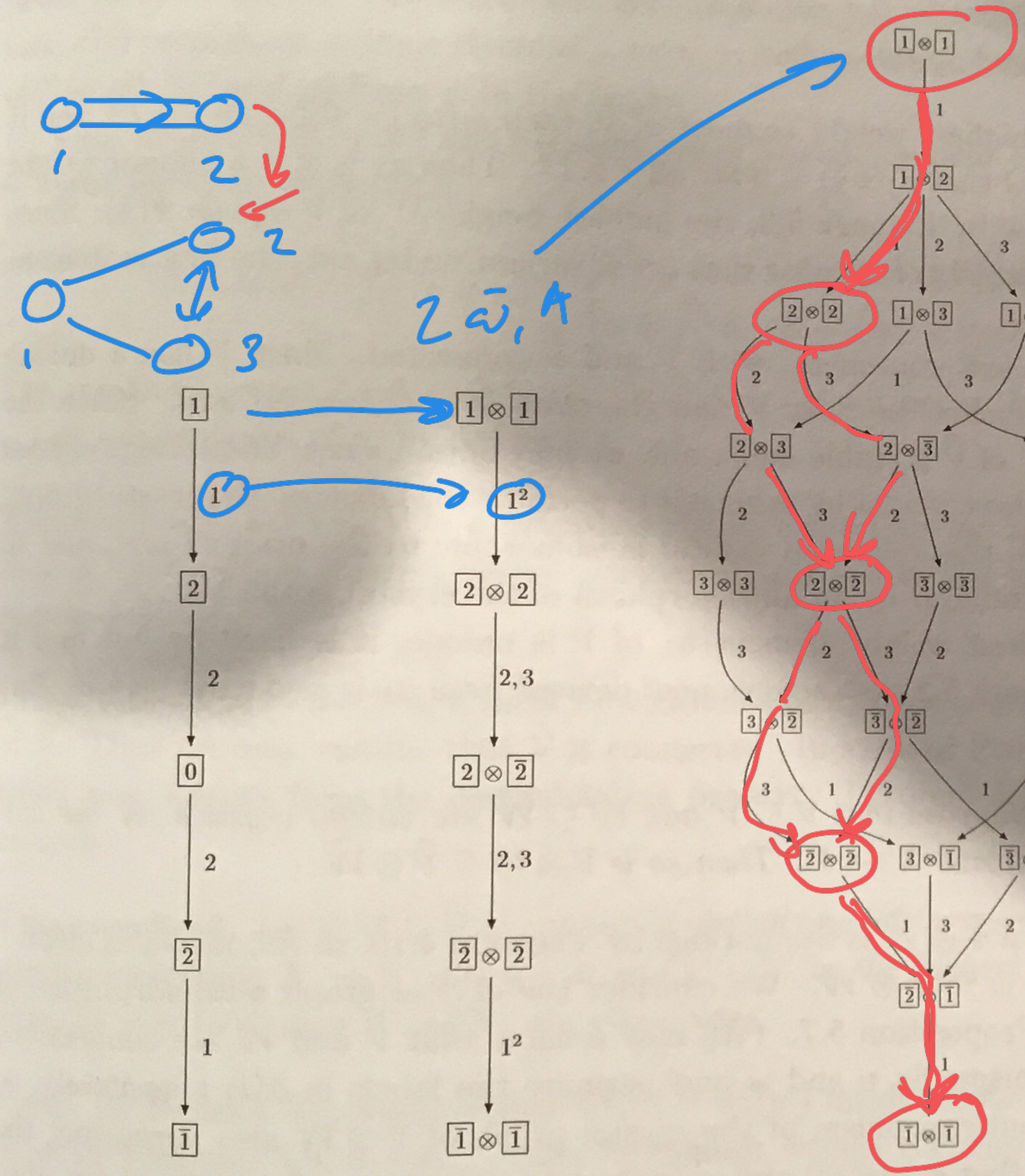


Fig. 5.4 The standard crystal of B_2 as a virtual crystal in D_3. Left: the B_2 crystal $\mathcal{B}_{\varpi_1}$. Right: one of three components in the D_3 crystal $\mathcal{B}_{\varpi_1} \otimes \mathcal{B}_{\varpi_1}$. Middle: the virtual crystal.