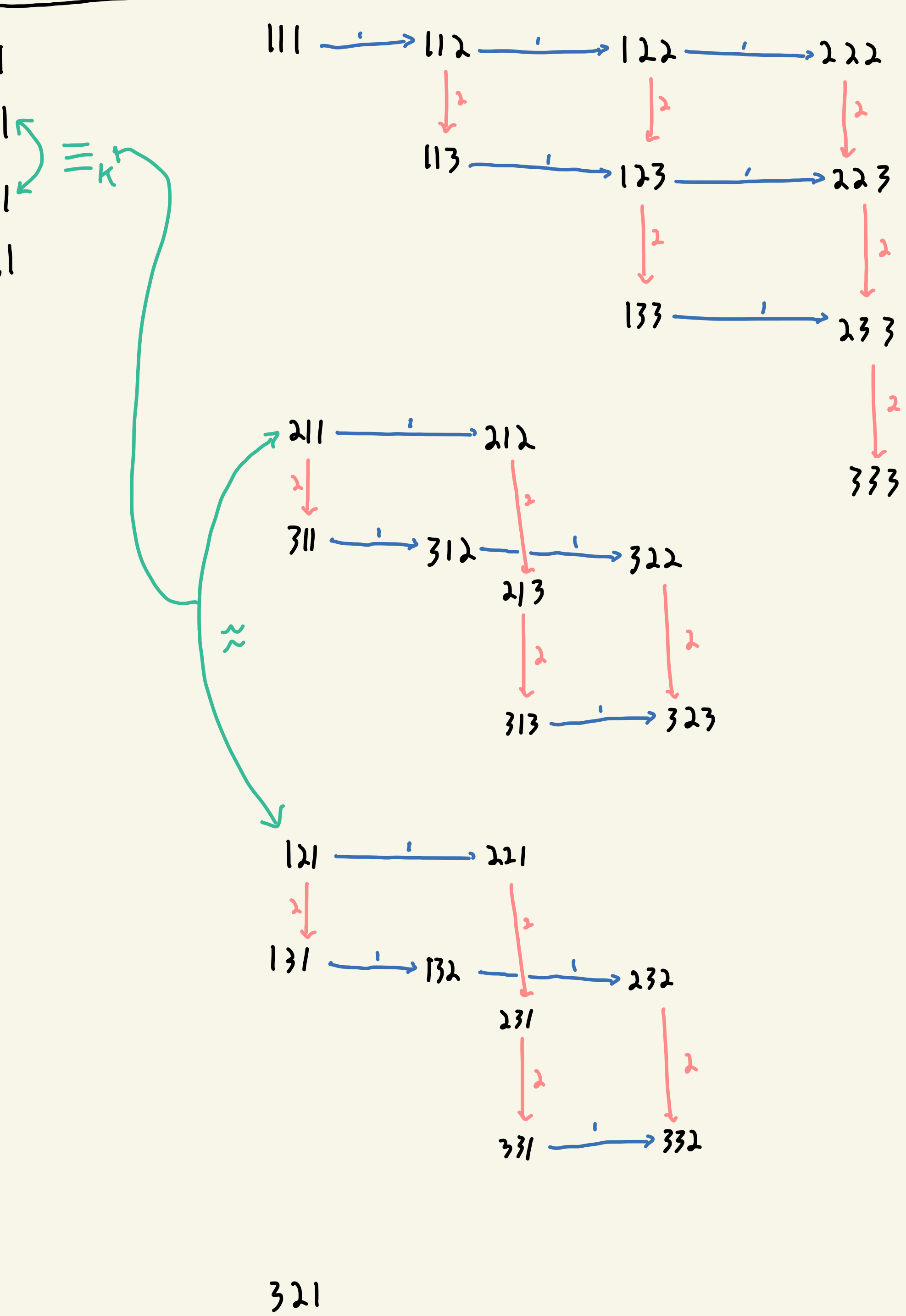
4/7: Examples of Knuth/Plactic equivalence

GL(3) crystal $\mathbb{B}\otimes\mathbb{B}\otimes\mathbb{B}$:

Yamanouchi words:

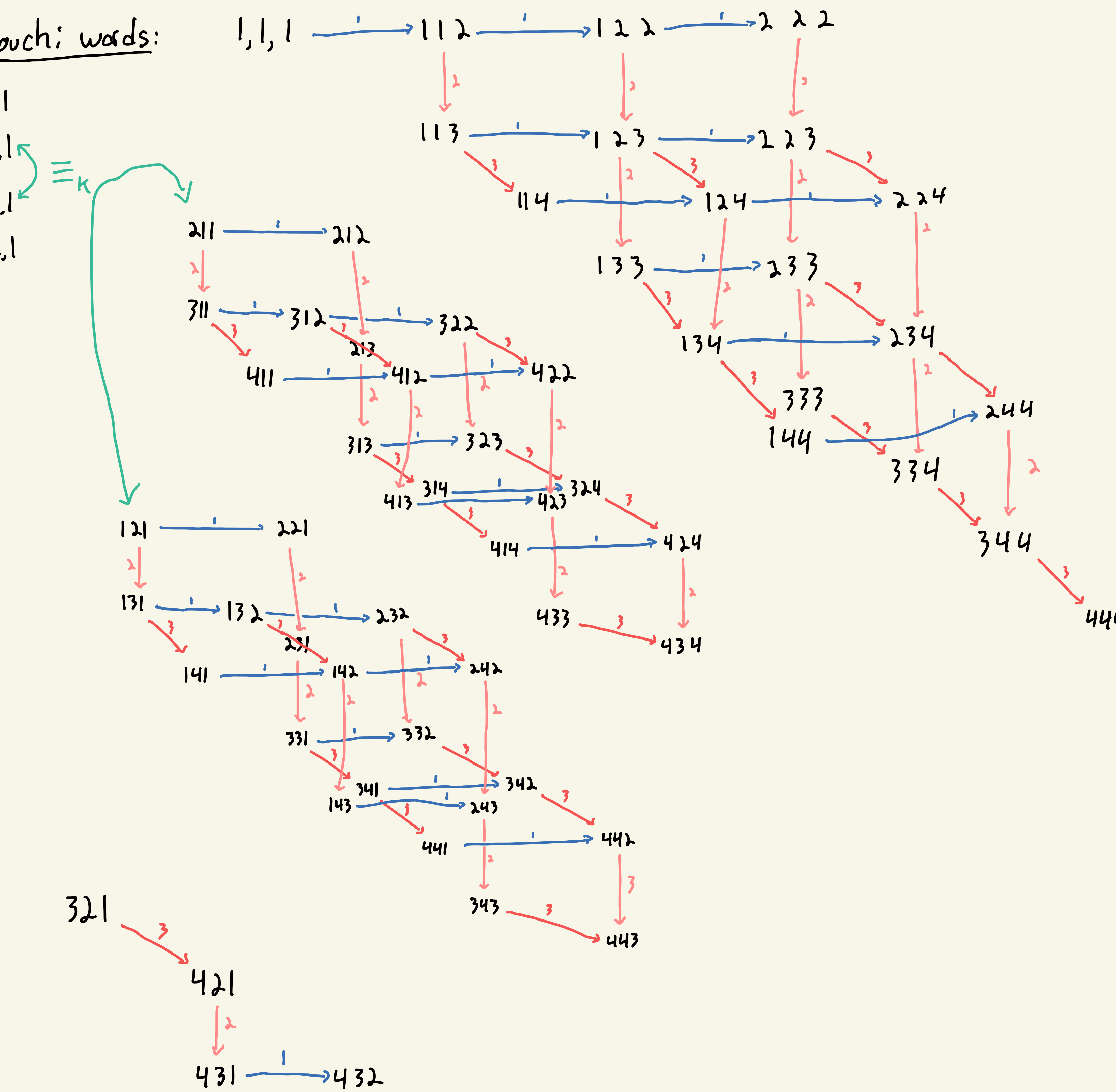
- 1,1,1
- 1,2,1
- 2,1,1
- 3,2,1



GL(4) crystal $\mathbb{B}\otimes\mathbb{B}\otimes\mathbb{B}$

Yamanouchi words:

- 1,1,1
- 1,2,1
- 2,1,1
- 3,2,1



GL(4) crystal $\mathbb{B}\otimes\mathbb{B}\otimes\mathbb{B}\otimes\mathbb{B}$

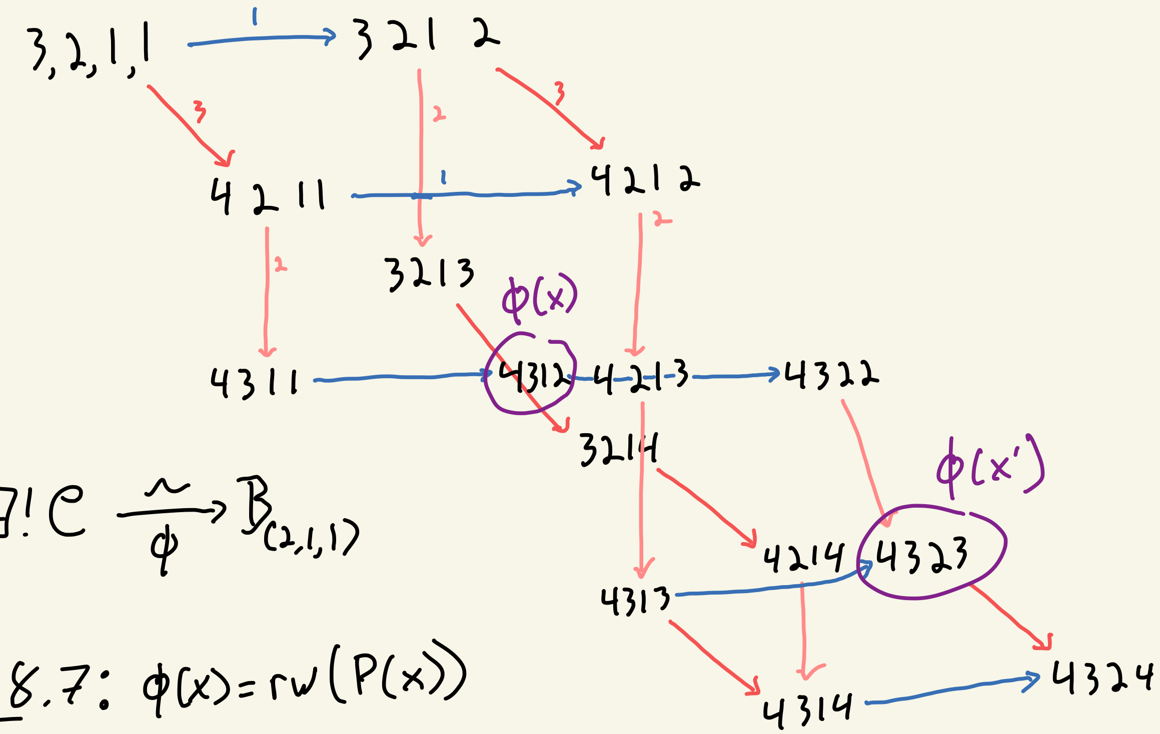
Questions:

- Which Yamanouchi words are Knuth equiv?
- Can we show plactic equiv. for these components?

Yamanouchi words:

- 1,1,1,1
- 2,1,1,1 $\equiv_K$ (type 1) 1,2,1,1 $\equiv_K$ (type 2) 1,1,2,1
- 2,2,1,1 $\equiv_K$ (type 2) 2,1,2,1
- 3,2,1,1 $\equiv_K$ (type 2) 3,1,2,1 $\equiv_K$ (type 2) 1,3,2,1
- 4,3,2,1

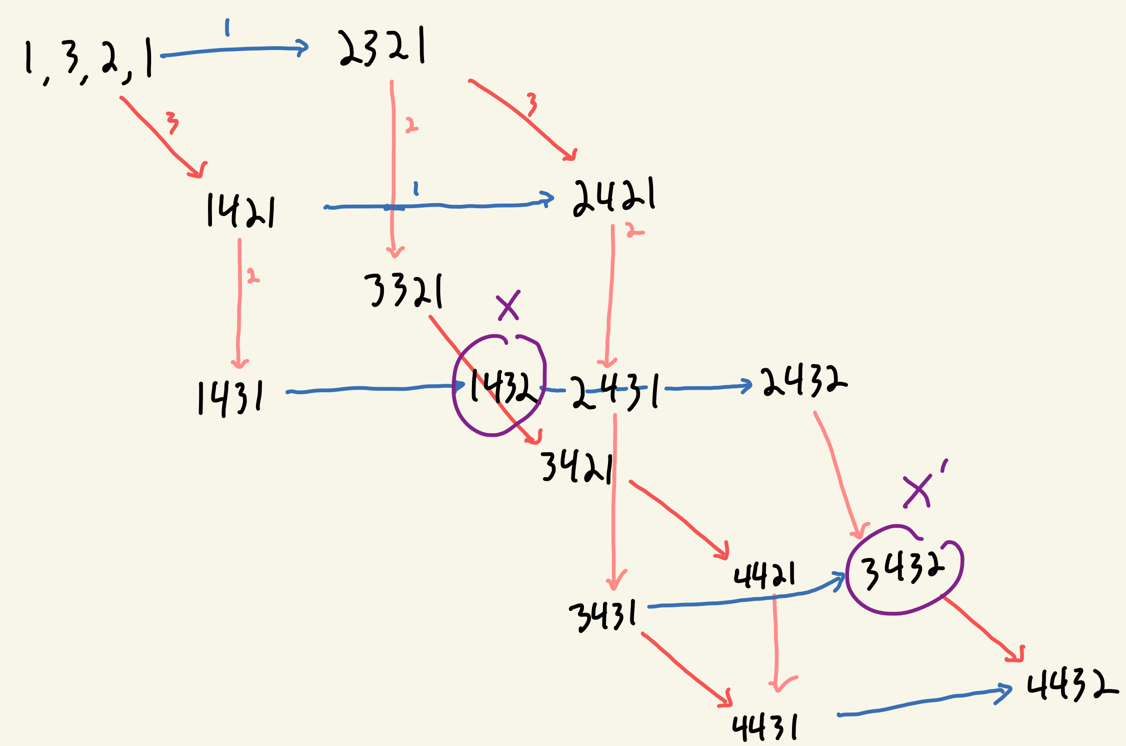
$B_{(2,1,1)}$ =



Stembridge: $\exists!\, \mathcal{C} \xrightarrow[\phi]{\sim} B_{(2,1,1)}$

Theorem 8.6.8.7: $\phi(x) = rw(P(x))$ & $Q(x) = Q(y)$ f $y \in \mathcal{C}$.

$\mathcal{C}$ =



Examples:

1) $x = 1432$, $\phi(x) = 4312$

$P(x)$ =

1	2
3	
4	

$Q(x)$ =

1	2
3	
4	

2) $x' = 3432$, $\phi(x') = 4323$

$P(x')$ =

2	3
3	
4	

$Q(x')$ =

1	2
3	
4	

Aside / interesting unanswered question

Q: Let $x \equiv_K y$ be Knuth equivalent Yamanouchi words, w/ isom ϕ: Component$(x) \xrightarrow{\sim}$ Component(y). If x, y are k Knuth moves apart, is the same true for all pairs $z, \phi(z)$? [Special case: $k=1$]

In example, is $\phi(x)$ always 2 Knuth moves from x? Can we deduce this from RSK?

Main Results of 8.1-8.3

Thm 8.6: $P(x)$ determines Plactic class, $rw(P(x)) \equiv x$

- Thm 8.4
- inductive insertion style argument

Thm 8.7: $Q(x)$ determines Connected Comp.

- Raising/Lowering ops don't affect $Q(x)$
- Consider sig. rule v.s. Schensted insertion
- "localizes" to $j, j+1$ argument

Excercise: -Verify both ↑ for small example: Compute $(P(x), Q(x))$ for all elts [perhaps in Parallel?]

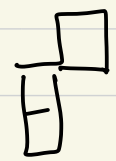
Next time:

- Ex 3.2 (schur Functions)
- General facts abt. schur functions
- Section 8.4
- More examples

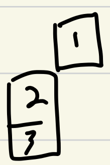
8.4: skew shapes have reading words, and these give crystals too!

Skew shapes:

Just like partitions, we have:

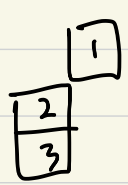
$\mu =$  , $\lambda =$  $\lambda/\mu =$ 

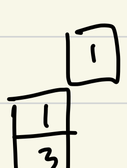
- Skew Tableau

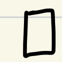
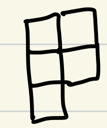
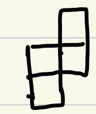


- Standard, Semistandard

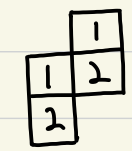
- Weight of a tableau

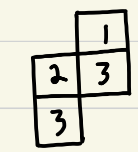
$wt\Big($  $\Big) = 1^1, 2^1, 3^1 \rightsquigarrow (1,1,1)$

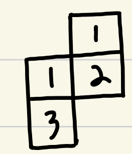
$wt\Big($  $\Big) = 1^2, 2^0, 3^1 \rightsquigarrow (2,0,1)$

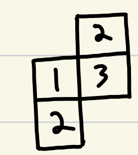
Ex $\mu =$  , $\lambda =$  , $\lambda/\mu =$ 

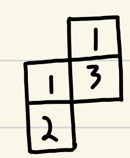
$SSYT_3(\lambda/\mu)$:

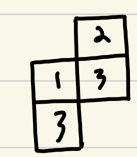
 $\xrightarrow{rw}$ 2 1 2 1 $\quad wt = (2,2,0)$

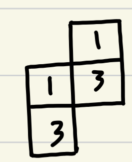
 $\xrightarrow{rw}$ 3 2 3 1 $\quad wt = (1,1,2)$

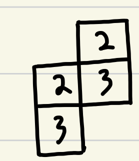
 $\xrightarrow{rw}$ 3 1 2 1 $\quad wt = (2,1,1)$

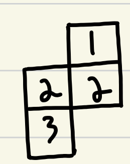
 $\xrightarrow{rw}$ 2 1 3 2 $\quad wt = (1,2,1)$

 $\xrightarrow{rw}$ 2 1 3 1 $\quad wt = (2,1,1)$

 $\xrightarrow{rw}$ 3 1 3 2 $\quad wt = (1,1,2)$

 $\xrightarrow{rw}$ 3 1 3 1 $\quad wt = (2,0,2)$

 $\xrightarrow{rw}$ 3 2 3 2 $\quad wt = (0,2,2)$

 $\xrightarrow{rw}$ 3 2 2 1 $\quad wt = (1,2,1)$

Littlewood-Richardson Coefficients

$$c^\lambda_{\mu\nu} = \#\left\{ \begin{array}{l} T \in SSYT(\lambda/\mu) \text{ with} \\ wt(T) = \nu \text{ and} \\ rw(T) \text{ a Yamanouchi word} \end{array} \right.$$

Theorem 8.8: For fixed n, the set $B_{\lambda/\mu} = SSYT_n(\lambda/\mu)$ is a crystal by identification w/

$$RR(B_{\lambda/\mu}) \subseteq \mathbb{B}^{\otimes|\lambda/\mu|}$$

Q: is this a full subcrystal? (A: seems so, not explicitly addressed)

Eg: $RR\left(\begin{smallmatrix} & 1 \\ 1 & 2 \\ 2 & \end{smallmatrix}\right) = 2 \otimes 1 \otimes 2 \otimes 1$

Prop 8.10 For large enough n, $B_{\lambda/\mu} \cong \bigoplus_{\nu} B_\nu^{\oplus c^\lambda_{\mu\nu}}$

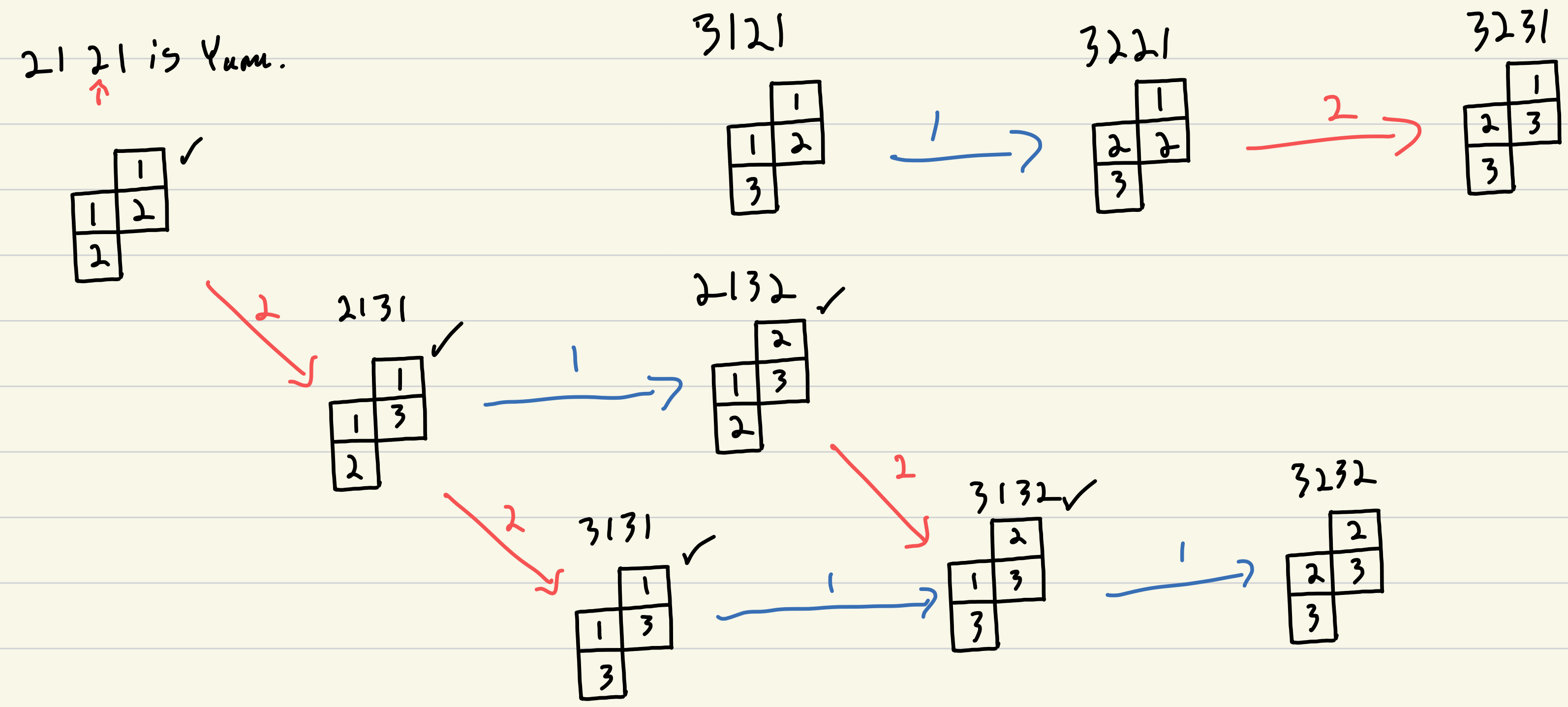
Book: $n \geq$ # of rows in λ/μ

Exercise 1

For $n=3$, $\mu = \square$, $\lambda = \begin{smallmatrix}\square&\square\\\square&\square\\\square&\end{smallmatrix}$,

Compute $B_{\lambda/\mu}$

Should be $\cong B_{2,1,1}$ ↓ (confirmed in book ✓)



↑ Should be $\cong$ to $B_{2,2}$ (confirmed in book ✓)

Exercise 2:

Find Yamanouchi words in 4 letters from $SSYT(\lambda/\mu)$

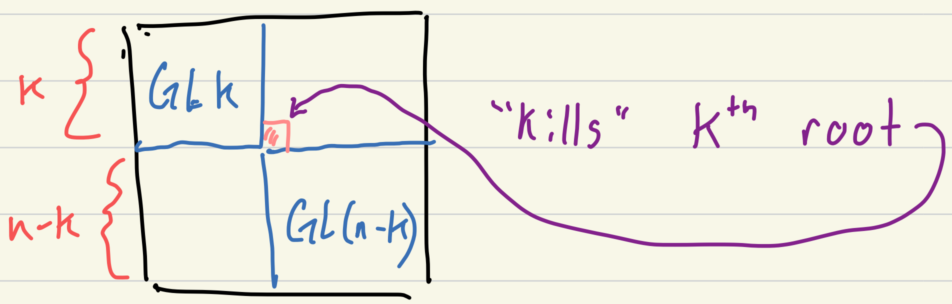
- 2,1,2,1 ✓
- 3,1,2,1 ✓

wt: 2,2   2,1,1

A second use of the L-R coefficients

"Recall" that $\boxtimes$ is the direct product corresponding to dir. prod. of root systems.

We can Levi Branch a $GL(n)$ crystal to a $GL(k)\times GL(n-k)$ crystal:



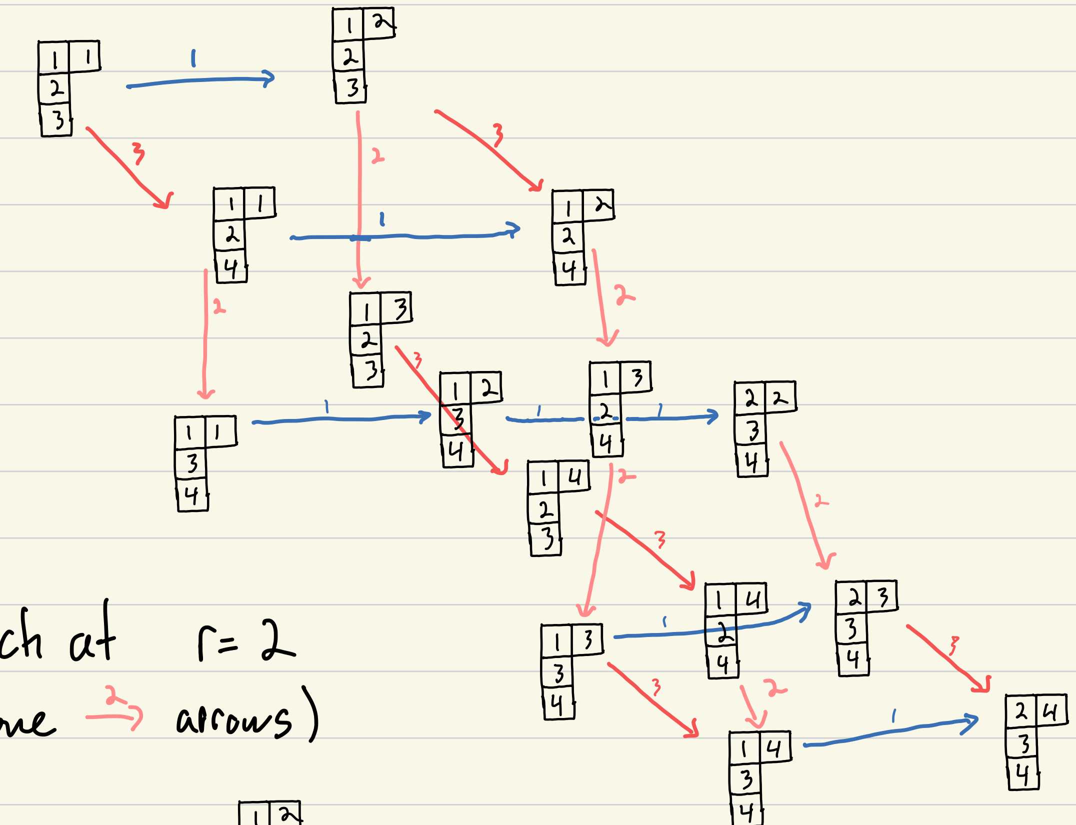
Theorem 8.14: For $\lambda \vdash k$, $\ell(\lambda)\le n$:

$$\mathrm{Branch}_{r,n-r}(B_\lambda) = \bigoplus_{\mu\subseteq\lambda} \underbrace{B_\mu}_{GL(r)} \boxtimes \underbrace{B_{\lambda/\mu}}_{GL(n-r)}$$

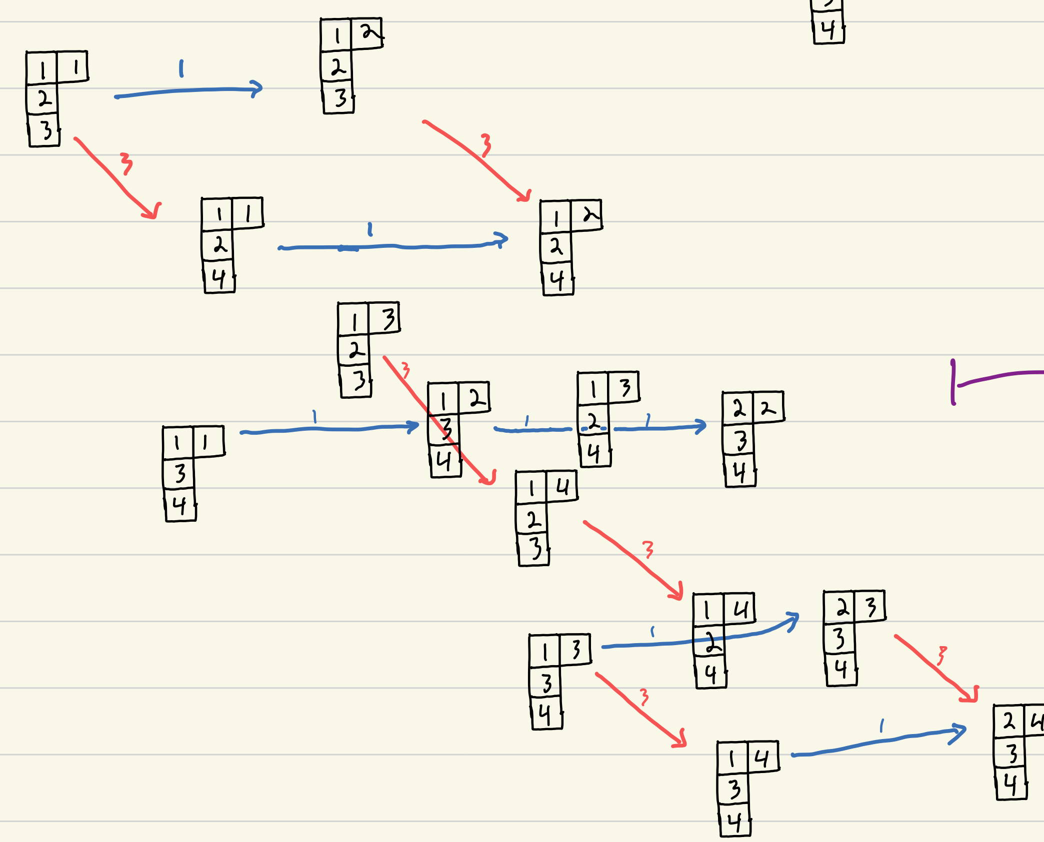
$$= \bigoplus_{\substack{\mu,\nu \\ |\mu|+|\nu|=k \\ \ell(\mu)\le n,\ \ell(\nu)\le n-r}} \left(B_\mu \boxtimes B_\nu\right)^{\oplus c^{\lambda}_{\mu\nu}}$$

Idea: μ is sub-tableau w/ entries $1,\dots,r$ and λ/μ is remainder

Example $n=4$, $\lambda = (2,1,1)$



Branch at $r=2$ (remove $\overset{2}{\to}$ arrows)



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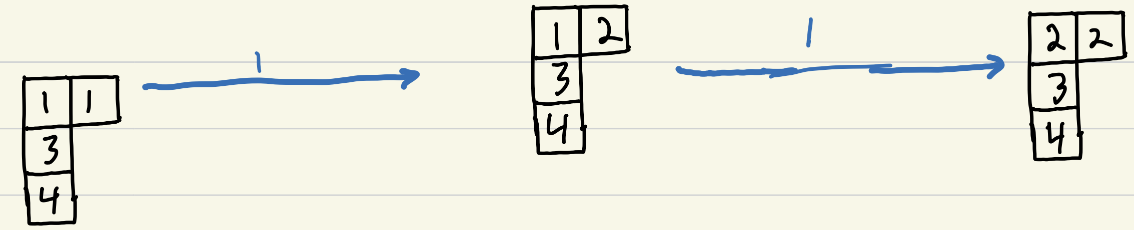
$\lambda = (2,1,1)$, $n = 4$,

$GL(4)$ branch to $GL(2) \times GL(2)$

$GL(4)$: A_3 $\qquad$ $GL(2)\times GL(2)$: $A_1 \times A_1$

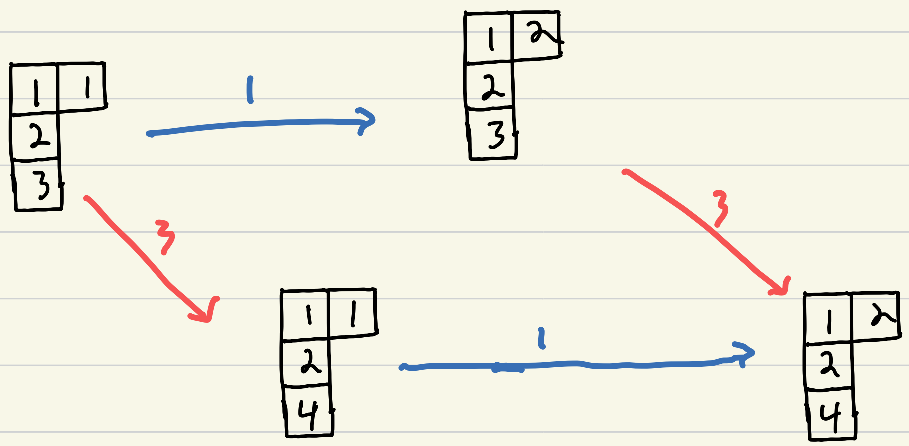
$B_{(2)} \boxtimes B_{(2,1,1)/(2)} =$

$B_{(2,1,1)/(2)} \approx B_{(1,1)}$



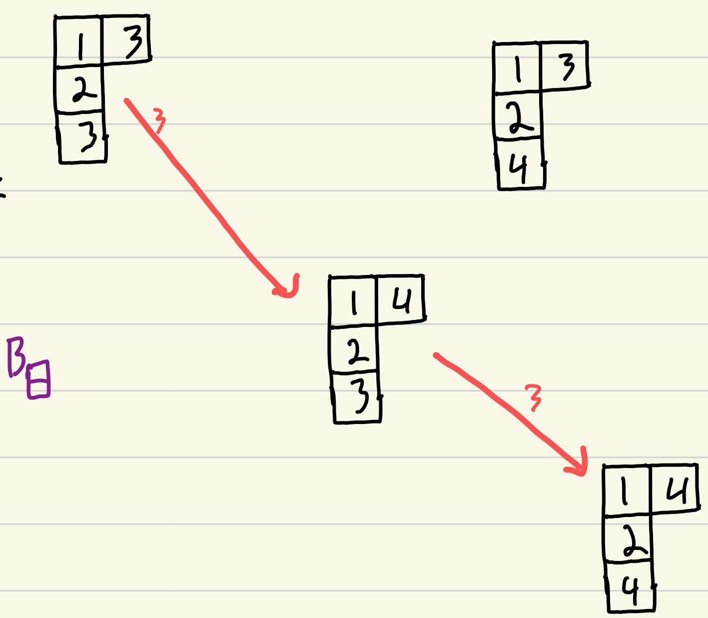
$B_{(2,1)} \boxtimes B_{(2,1,1)/(2,1)} =$

$B_{(2,1,1)/(2,1)} \approx B_{(1)}$



$B_{(1,1)} \boxtimes B_{(2,1,1)/(1,1)} =$

$B_{(2,1,1)/(1,1)} \approx B_{(2)} \oplus B_{(1,1)}$



$B_{(1)} \boxtimes B_{(2,1,1)/(1)} =$

$B_{(2,1,1)/(1)} \approx B_{(2,1)}$

*on 2 letters