Chapter 2. Kashiwara Crystals

(**Ref:**

Introduction to Lie Algebras. Erdmann & Wildon

Introduction to Lie Algebra and Representation Theory. Humphreys)

2.1. Root Systems.

Part I. Roots

Definitions.

1. Euclidean space: real vector space w/ inner product, i.e., with

V, $\langle \, , \, \rangle = (\, , \,)$ a positive definite, symmetric bilinear form.

Prototypical example: $\mathbb{R}^d$, $\langle \, , \, \rangle$ usual inner product.

2. Reflection maps:

Euclidean space V, $\langle \; \rangle$ $\}$ $\rightarrow$ Hyperplane H_α orthogonal to α

α nonzero

$$H_\alpha = \ker \langle -, \alpha \rangle$$

$$\Downarrow$$

reflection $r_\alpha : V \rightarrow V$ across H_α.

formula $\beta \longmapsto \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha$.

About the formula for r_α : it sends α to $-\alpha$ and fixes everything $\perp$ to α, so it's correct.

Shorthand. write $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ for, so $r_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$.

$\frac{2(\beta,\alpha)}{(\alpha.\alpha)}$

if α is a 'root', call $\alpha^\vee$ its coroot.

Note: ① if $\langle\alpha,\alpha\rangle = 2$. eg. $\alpha = e_i - e_j \in \mathbb{R}^d$,

then $\alpha^\vee = \alpha$. ② $(\alpha^\vee)^\vee = \frac{2\alpha^\vee}{(\alpha^\vee,\alpha^\vee)} = \frac{2\frac{2\alpha}{(\alpha,\alpha)}}{\left(\frac{2\alpha}{(\alpha,\alpha)}, \frac{2\alpha}{(\alpha,\alpha)}\right)} = \alpha$.

involution.

Note: Reflections are isometries.

$$\left(\beta_1 - \frac{2(\beta_1,\alpha)}{(\alpha,\alpha)}\alpha, \beta_2 - \frac{2(\beta_2,\alpha)}{(\alpha,\alpha)}\alpha\right) = (\beta_1,\beta_2) - \cancel{\frac{4(\alpha,\beta_2)(\alpha,\beta_1)}{(\alpha,\alpha)}} + \cancel{\frac{4(\beta_1,\alpha)(\beta_2,\alpha)}{(\alpha,\alpha)}}$$

$$= (\beta_1,\beta_2)$$

Def of root systems. A root system Φ in V is a nonempty, finite set of nonzero vectors s.t.

① $r_\alpha(\Phi) = \Phi$ $\forall \alpha \in \Phi$, i.e., $r_\alpha(\beta) = \beta - \underbrace{\frac{2(\beta,\alpha)}{(\alpha,\alpha)}}_{\in \mathbb{Z}} \alpha \in \Phi$

$\forall \alpha, \beta \in \Phi$.

② $(\alpha, \beta^\vee) \in \mathbb{Z}$ $\forall \alpha, \beta \in \Phi$. i.e., $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ $\forall \alpha, \beta \in \Phi$.

③ If $\beta \in \Phi$ is a multiple of $\alpha \in \Phi$, then $\beta = \alpha$ or $\beta = -\alpha$.

Each $\alpha \in \Phi$ is called a root.

Ex. Φ is a root system $\Longrightarrow$ $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ is also a root system

Pf: ③ $\checkmark$ easy. ② $(\alpha^\vee, (\beta^\vee)^\vee) = (\alpha^\vee, \beta) \in \mathbb{Z}$

① $r_{\alpha^\vee}(\beta^\vee) = r_\alpha(\beta^\vee) = r_\alpha\left(\frac{2\beta}{(\beta,\beta)}\right) = \frac{2r_\alpha(\beta)}{(\beta,\beta)}$

$\Phi^\vee \ni (r_\alpha(\beta))^\vee = \frac{2r_\alpha(\beta)}{(r_\alpha(\beta), r_\alpha(\beta))} = \frac{2r_\alpha(\beta)}{(\beta,\beta)}$ //

More defn.

Later: $A_1 \times A_1$ is not reducible

- Φ is reducible if it's the union of two proper, orthogonal subsets which are themselves root systems. Otherwise Φ is irreducible or simple.

- Φ is simply-laced if all roots have the same length.

How does this tie into our familiar notion of simply-laced for Coxeter graphs? Answer: later.

Correspond to the analogous notion for Coxeter gps.

Consequences of the root system axiom I.

1. Let $\alpha, \beta \in \Phi$, $\beta \neq -\alpha$. Then $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}$.

Angles

($\langle \, , \rangle$ is not symm in the arguments)

Pf: $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = \dfrac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \quad \dfrac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 4 \dfrac{\|\alpha\|^2 \|\beta\|^2 \cos^2\theta}{\|\alpha\|^2 \|\beta\|^2}$

$= 4\cos^2\theta \in \mathbb{Z} \cap \{0, 1, 2, 3, 4\}$, where θ is the angle between α and β.

$4\cos^2\theta = 4 \Rightarrow \cos\theta = \pm 1 \Rightarrow \alpha = \pm\beta$, $\rightarrow\leftarrow$, so $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}$.

In fact, if we assume $\|\beta\| \geq \|\alpha\|$, there are only finitely many possibilities. $\left(|\langle \beta, \alpha^\vee \rangle| = \dfrac{|2(\beta, \alpha)|}{\|\alpha\|^2} \geq \langle \alpha, \beta^\vee \rangle = \dfrac{|2(\alpha, \beta)|}{\|\beta\|^2} \right)$

$4\cos^2\theta = \langle\alpha,\beta^\vee\rangle\langle\beta,\alpha^\vee\rangle$	$\langle\alpha,\beta^\vee\rangle$ ✓ $\in\mathbb{Z}$	$\langle\beta,\alpha^\vee\rangle$ $\in\mathbb{Z}$,	θ	✓ $\frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle}$	$r_\beta(\alpha) = \alpha - \langle\alpha,\beta^\vee\rangle\beta$.	
0	0	0	$\frac{\pi}{2}$	undetermined.	α	
1	1	1	$\frac{\pi}{3}$	1	$\alpha-\beta$	
1	-1	-1	$\frac{2\pi}{3}$	1	$\alpha+\beta$	$\alpha \overset{(3)}{-} \beta$
2	1	2	$\frac{\pi}{4}$	2	$\alpha-\beta$	↓ simple e e
2	-1	-2	$\frac{3\pi}{4}$	2	$\alpha+\beta$	$\beta \Rightarrow \alpha$
3	1	3	$\frac{\pi}{6}$	3	$\alpha-\beta$	↓ double bond
3	-1	-3	$\frac{5\pi}{6}$	3	$\alpha+\beta$	$\beta \Rrightarrow \alpha$ triple bond

Corollary: Let $\alpha, \beta \in \Phi$ and let θ be the angle between them. Assume $\|\beta\| \geq \|\alpha\|$

① if $\theta > \frac{\pi}{2}$ then $\alpha+\beta \in \Phi$ ② if $\theta < \frac{\pi}{2}$ then $\alpha-\beta \in \Phi$.

2. **Thm.** (a) Every root system has a base, that is, a set $\Sigma = \{\alpha_i \mid i \in I\}$

Base.

s.t. ① Σ is linearly ind. and span Φ.

② every $\beta \in \Phi$ can be written as $\beta = \sum_{i \in I} c_i \alpha_i$

s.t. either $\underbrace{c_i \geq 0 \ \forall i \in I}_{\text{positive roots. } \Phi^+}$ or $\underbrace{c_i \leq 0 \ \forall i \in I}_{\text{negative roots } \Phi^-}$.

Elements of Σ are called the simple roots.

Note: if $\alpha, \beta \in \Sigma$, then $\langle \alpha, \beta \rangle < 0$. i.e. $\theta_{\alpha,\beta} > \frac{\pi}{2}$. (see earlier corollary)

(b) One way to get a base: Choose $z \in V$ not perpendicular to any roots, let $\Phi_z^+ = \{\alpha \in \Phi : (\alpha, z) > 0\}$ and let $\Sigma = \{\alpha \in \Phi_z^+ : \alpha \text{ is not a sum of two elts in } R^+\}$. Then Σ is a base of Φ with $\Phi^+ = \Phi_z^+$. In particular, a base of Φ always exists but is not unique.

(c) Pick a base Σ of Φ and a simple root α_i. Write $s_i = r_{\alpha_i}$.

Then ① On Φ^+ s_i sends α_i to $-\alpha_i$

and permutes $\Phi^+ \setminus \{\alpha_i\}$.

② The Weyl gp of Φ, the gp W gen. by $\{r_\alpha : \alpha \in \Phi\}$, is generated by $\underbrace{\{s_i : i \in I\}}_{\text{simple reflections}}$. In fact,

there are relations among them automatically.

long → short

W is a finite Coxeter gp with $\{s_i : i \in I\}$ as its Coxeter generators.

$s_i^2 = 1$. ✓

reflection.

$$(s_i s_j)^{m_{ij}} = 1.$$

ref ref

order $= \frac{2\pi}{2\gamma} = \begin{cases} 3 \\ 4 \\ 6 \end{cases}$

Dynkin (see table)

simple edge

double bond

triple bond

rotation by 2γ, γ : acute angle between $\mathbb{R}\alpha_i$, $\mathbb{R}\alpha_j$

Note: $W(\Phi) = W(\Phi^\vee)$

so $W(B_n) = W(C_n)$ redundant

③ From any choice of base Σ for Φ, we may define an $I \times I$ matrix C called the Cartan matrix of Φ, with $C_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$.

(so $C_{ii} = 2 \ \forall \ i \in I$ and $\{C_{ij}, C_{ji}\} \in \{ \{-1,-1\}, \{-1,-2\}, \{-1,-3\}, \{0\} \}$).

interaction of simple roots

Up to reordering of rows and columns, C is independent of the choice of the base, so it's well-defined.

encodes all info in the base (all info in Lie alg)

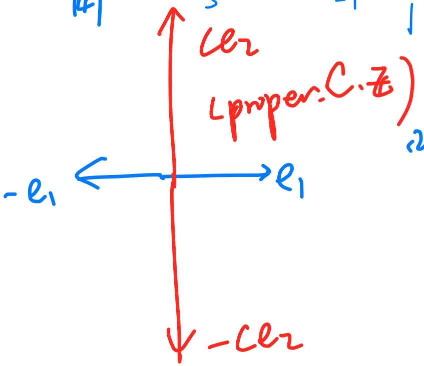
④ It turns out that using the fact that a root system is finite by definition, we can classify all root systems by their Dynkin diagram or Cartan types. The types are: A_n, $\underbrace{B_n, C_n}_{\text{dual}}$, D_n, E_6, E_7, E_8, F_4, G_2.

We can work out some possible root systems in $\mathbb{R}^2$ but not $\mathbb{R}$:

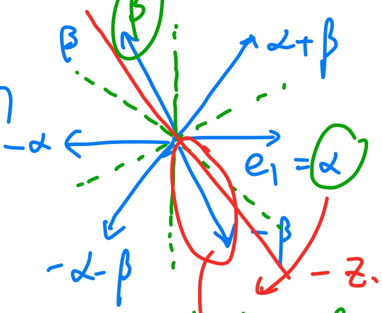
- $|\Sigma| \le \dim \mathbb{R}^2 = 2$. So Φ is 2-dim $\Longrightarrow |\Sigma| = 2$. say $\Sigma = \{\alpha, \beta\}$
- Recall the possibilities assuming $\|\beta\| \ge \|\alpha\|$. May assume $\alpha = e_1$:

	$\langle\alpha,\beta\rangle\langle\beta,\alpha^\vee\rangle$	$\langle\alpha,\beta^\vee\rangle$	$\langle\beta,\alpha^\vee\rangle$	θ	$\frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle}$	$r_\beta(\alpha)=\alpha-\langle\alpha,\beta^\vee\rangle\beta$.	
(1)	0	0	0	$\frac{\pi}{2}$	undetermined.	α	
(2)	1	-1	-1	$\frac{2\pi}{3}$	1	$\alpha+\beta$	$\alpha-\beta^{(3)}$
(3)	2	-1	-2	$\frac{3\pi}{4}$	2	$\alpha+\beta$	$\beta=$
(4)	3	-1	-3	$\frac{5\pi}{6}$	3	$\alpha+\beta$	$\overset{6}{\beta}\Rrightarrow\alpha$

(1):



(2)



$\to \Sigma_Z = \{\alpha, \beta\}$.

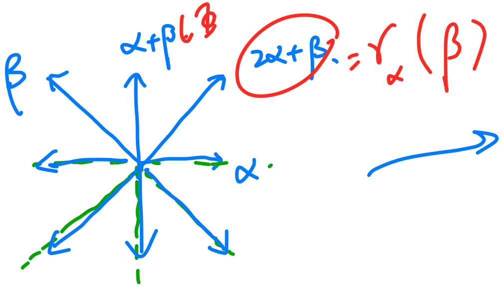
$\longrightarrow A_2$

α, β are closest to CZ.

green: the hyperplanes

$-\beta = r_{\alpha+\beta}(\alpha) \in \Phi$

(3)



$\Longrightarrow$ B_2

when $r = 2$, Φ lies in the hyperplane orthogonal to $(1,1,1)$ in $\mathbb{R}^3$.

hyperplane ↳ 2D!

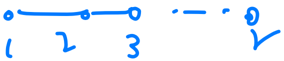
$$\cos\theta(\alpha_1,\alpha_2) = \frac{(\alpha_1,\alpha_2)}{\|\alpha_1\|\|\alpha_2\|} = \frac{-1}{2} \Rightarrow \theta = \frac{2}{3}\pi$$

similarly we can compute $\theta(\alpha,\beta)$ $\forall \alpha,\beta$.

↑ Then we can show $\Phi \cong \text{Ex}(2)$.

✡ (4) G_2. Exercise.

(5) Exercise: Check that $\Phi := \{\pm(\underset{\alpha}{e_i - e_j}) : 1 \le i < j \le r+1\}$ $\overset{(\alpha,\alpha)=2}{}$ forms A^r a root system in $V = \mathbb{R}^{r+1}$, and $\Sigma := \{\underbrace{e_i - e_{i+1}}_{\alpha_i}\} \subset \Phi$ is a base of Φ. The Dynkin diagram of Φ is



$C_{13} = (e_1 - e_2, e_3 - e_4)/(\ \ ) = 0$

$\circlearrowleft \mathbb{R}^{r+1}$

Φ doesn't span V, so Φ is not semisimple.

and $S_i = r(\alpha_i)$ 'permutes e_i and e_{i+1}' so $W(\Phi) \cong S_{r+1}$.

$\{S_i : i \in I\} \circlearrowleft \{e_j : j \in [r+1]\}$

$$S_i(e_j) = \underline{e_j - \langle e_j, e_i - e_{i+1}\rangle(e_i - e_{i+1})} = \begin{cases} e_j & \text{if } j \notin \{i, r+1\} \\ e_{r+1} & \text{if } j = i \\ e_i & \text{if } j = i+1. \end{cases}$$

Part II. Weights

Definition 1.

- Given a root system Φ in an Euclidean space V, a weight lattice is a lattice Λ spanning V s.t. $\Phi \subset \Lambda$ and $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ $\forall \lambda \in \Lambda, \alpha \in \Phi$.

  Elements of Λ are called weights.

- A wt. lattice $\Lambda \subset V$ is called semisimple if Φ spans V.

(the condition is not about Λ per se! Root systems coming from Lie algebras are often/can often be assumed to be semisimple.)

? "This is equivalent to assuming that the root lattice Λ_{root} spanned by Φ has finite codimension in Λ." — there will be examples.

— There's a partial order on Λ: for $\lambda, \mu \in \Lambda$. write

$$\lambda \geq \mu \quad \text{if} \quad \lambda - \mu = \sum_{i \in I} c_i \alpha_i \quad \text{where} \quad c_i \geq 0 \ \forall i \in I.$$

— (Dominant wts) Let $\Lambda_+ = \{ \lambda \in \Lambda : \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ \ \forall \ i \in I \}$

Elts of Λ_+ are called dominant wts. $\lambda \in \Lambda_+$ is strictly dominant if $\langle \lambda, \alpha_i^\vee \rangle > 0 \ \forall \ i \in I$.

Common expression $(- , \alpha^\vee)$ 'eigenvalues of h_α on a module'

— (Fundamental wts) $\forall i \in I$, $\exists \ \underline{\varpi_i} \in V$ s.t. fundamental wts

inner product $(\ , \ )$ ↓ identify V w/ V^*:

(*) $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$.

If Φ is semisimple (span V), (*) determines ϖ_i; otherwise there are choices for ϖ_i.

↓ (bilinear form argument).

Note: fundament wts may not be in the wt lattice !

— Assuming $\Phi \subset \Lambda \subset V$ is semisimple. Then the fundamental weights generate a lattice Λ_{sc} that contains Λ_{root} as a sublattice of finite index. We have

$$\Lambda_{sc} \supseteq \Lambda \supseteq \Lambda_{root}.$$

↘

$\lambda \in \Lambda$, $\langle \lambda, \alpha_i \rangle = c_i \geq 0$ $\forall i \in I$ $\Longrightarrow$ $\lambda = \sum c_i \tilde{\omega}_i \in \Lambda_{sc}$.

If $\Lambda = \Lambda_{root}$, we say Λ is of adjoint type.

wts are exactly roots

(comes from adjoint rep)

If $\Lambda = \Lambda_{sc}$, i.e., if all fundamental wts are in the wt lattice, we say Λ is of simply-connected type. (← can often enlarge Λ to be simply-connected)

(motivation?)

not semisimple. $\Phi \perp e_1 + - + e_{r+1}$.

can also pick $\varpi_i' = \varpi_i + (e_1 + \cdots + e_{r+1})$.

Examples.

(1) Type A_r. (BS Ex. 2.5. 'GL(r+1) repsn')

$V = \mathbb{R}^{r+1}$. $\quad \Phi = \{ e_i - e_j \mid 1 \le i, j \le r+1 \}$. $\quad \Sigma = \{ e_i - e_{i+1} \mid 1 \le i \le r \}$

$\Phi^+ = \{ e_i - e_j \mid 1 \le i < j \le r+1 \}$.

Take $\Lambda = \mathbb{Z}^{r+1} = \{ \underset{\overset{\|}{\lambda}}{(\lambda_1, \dots, \lambda_{r+1})} : \lambda_i \in \mathbb{Z} \ \forall i \}$

Note: $\langle \lambda, (e_i - e_j)^\vee \rangle = \langle \lambda, e_i - e_j \rangle = \lambda_i - \lambda_j \in \mathbb{Z} \ \forall \lambda \in \Lambda$.

and of course Λ span V.

$\lambda \in \Lambda^+ \iff (\lambda, e_i - e_{i+1}) \ge 0 \ \forall i \iff \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{r+1}$

fundamental wts: $\varpi_i = e_1 + \cdots + e_i$ satisfies $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$. so

We can pick ϖ_i to be the fundamental wt.

(2) (Semisimplication of (1). Type A_r, $SL(r+1)$ version)

$V = \mathbb{R}^{r+1}/\mathbb{C}\langle(1,\dots,1)\rangle$. Φ = Image of Φ from (1) in the quotient V.

Λ = image of Λ from (1) in the quotient.

(3). (B_r, $Spin(2r+1)$) $V = \mathbb{R}^r$, $\Phi = \{\underline{\pm e_i \pm e_j} \mid i<j\} \cup \{\underline{\pm e_i}\}$

not simply-laced $\dfrac{\|\text{long}\|^2}{\|\text{short}\|^2} = 2$

* $\Phi^+ = \{e_i \pm e_j \mid i<j\} \cup \{e_i\}$.

$\Sigma = \{e_1-e_2, \dots, e_{r-1}-e_r, e_r\}$.

$\Lambda = \{\underset{\|}{(\lambda_1,\dots,\lambda_r)} \in \mathbb{Q}^r \mid 2\lambda_i \in \mathbb{Z}\ \forall i,\ 2\lambda_i \text{ are either all odd or all even}\}$

Note: $\langle\lambda, e_i^\vee\rangle = 2\lambda_i$, $\langle\lambda, e_i \pm e_j\rangle = \lambda_i \pm \lambda_j$. so $\langle\Lambda, \alpha^\vee\rangle \in \mathbb{Z}$ $\forall \alpha \in \Phi$.

all in Λ, so simply-connected!

↖

dominant: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$.

fund. wts: $\varpi_1 = e_1$, $\varpi_2 = e_1 + e_2$, $\dots$, $\varpi_{r-1} = e_1 + \dots + e_{r-1}$ $\varpi_r = \frac{1}{2}(e_1 + \dots + e_r)$.

Exercises. (a) Check Ex 2.8.

Φ as in $(B_r, \mathrm{Spin}(2r+1))$

$\Lambda = \Lambda_{\mathrm{root}} = \mathbb{Z}^r$, not simply-connected.

(b) Check Ex 2.9. C_r. everything is dual to $(B_r, \mathrm{Spin}(2n+1))$.

(c) Compute the type B_r and C_r Weyl gps

(They should be the same and iso to the Coxeter gp of type B.)

Part III. Lie algebraic origins of roots and weights

Roughly speaking,

- every (finite dimensional) Lie algebra is made up of so-called 'sl_2-triples'. How these triples are organized can be encoded by root systems, so root systems encode the internal structure of Lie algebras.

- every Lie algebra L has a subalgebra H such that for any repn V of L, $h \cdot v = \lambda(h) v$ for some $\lambda(h) \in \mathbb{C}$ $\forall h \in H$, $v \in V$. that is, the entire H acts 'diagonally' on V. The information of what scalars elts of H act as will be encoded by our weights.

— It turns out that a repn V of a Lie algebra is completely determined by what the associated weights are, so weights are sufficient to encode representations.

— By the above, to encode the action of a Lie algebra L on a repn V, we need to use roots to encode the internal structure of L and use wts to encode the structure of V and the L-action on V.

Trying to do so leads to crystals.

Facts about reps of sl_2-triples and relations among the sl_2-triples in L will translate naturally to restrictions/axioms on roots/wts/crystals

Some translation!

Roots ⟷ linear functionals $\alpha : H \to \mathbb{C}$ on the subalg H.

$\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ $\forall \lambda \in \Lambda, \alpha \in \Phi$ ⟷ $\lambda(h_\alpha)$, where $\{h_\alpha, e_\alpha, f_\alpha\}$ is a 'sl_2-triple' in L, must be an integer by sl_2-theory

$$\Big(\quad (\lambda, \alpha^\vee) \quad = \quad \lambda(h_\alpha) \quad \Big)$$

Crystal axiom

$$wt(e_\alpha v) = wt(v) + \alpha$$

⟷ somehow

old eigenvector

if $h_\alpha \cdot \boxed{v} = \lambda \boxed{v}$, then

$$h_\alpha \boxed{(e_\alpha \cdot v)} = \underbrace{(\lambda+2)\boxed{(e_\alpha \cdot v)}}_{\text{new eigenvector}}$$

This is lin alg:

$$h_\alpha e_\alpha v = (e_\alpha h_\alpha + 2e_\alpha)\, v = (\lambda+2)(e_\alpha \cdot v)$$

since $[h_\alpha, e_\alpha] = 2e_\alpha$. □

point: since v and $e_\alpha v$ are related by e_α and e_α interacts with elts of H via relations in L, wts on $e_\alpha v$ and on v are rel. by roots.