Chapter 2. Kashiwara Crystals

(Ref:

Introduction to Lie Algebras. Erdmann & Wildon

Introduction to Lie Algebra and Representation Theory. Humphreys)

2.1. Root Systems.

Part I. Roots

Definitions.

1. Euclidean space : real vector space w/ inner product, i.e., with

   $V, \langle\, , \,\rangle$. a positive definite, symmetric bilinear form.

Prototypical example : $\mathbb{R}^d$, $\langle\, , \,\rangle$ usual inner product.

2. Reflection maps :

Euclidean space V, $\langle \ \rangle$ } $\rightarrow$ Hyperplane H_α orthogonal to α

α non zero

$$H_\alpha = \ker \langle -, \alpha \rangle$$

$$\Downarrow$$

reflection $r_\alpha : V \rightarrow V$ across H_α.

formula $\beta \longmapsto \beta - \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha$.

About the formula for r_α : it sends α to $-\alpha$ and fixes everything $\perp$ to α, so its correct.

Shorthand write $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$, so $r_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$.

↙

if α is a 'root', call $\alpha^\vee$ its coroot.

Note: ① if $\langle \alpha, \alpha \rangle = 2$. eg. $\alpha = e_i - e_j \in \mathbb{R}^d$, then $\alpha^\vee = \alpha$. ② $(\alpha^\vee)^\vee = \frac{2\alpha^\vee}{(\alpha^\vee, \alpha^\vee)} = \frac{2\frac{2\alpha}{(\alpha,\alpha)}}{\left(\frac{2\alpha}{(\alpha,\alpha)}, \frac{2\alpha}{(\alpha,\alpha)}\right)} = \alpha$.

involution.

Note: Reflections are isometries.

$$\left(\beta_1 - \frac{2(\beta_1, \alpha)}{(\alpha, \alpha)}\alpha, \beta_2 - \frac{2(\beta_2, \alpha)}{(\alpha, \alpha)}\alpha\right) = (\beta_1, \beta_2) - \cancel{\frac{4(\alpha, \beta_2)(\alpha, \beta_1)}{(\alpha, \alpha)}} + \cancel{\frac{4(\beta_1, \alpha)(\beta_2, \alpha)}{(\alpha, \alpha)}}$$

$$= (\beta_1, \beta_2)$$

Def of root systems. A root system Φ in V is a nonempty, finite set of nonzero vectors s.t.

① $r_\alpha(\Phi)=\Phi$ $\forall \alpha\in\Phi$, i.e., $r_\alpha(\beta)=\beta-\underbrace{\frac{2(\beta,\alpha)}{(\alpha,\alpha)}}_{\in\mathbb{Z}.}\alpha\in\Phi$ $\forall \alpha,\beta\in\Phi$.

② $(\alpha,\beta^\vee)\in\mathbb{Z}$ $\forall\alpha,\beta\in\Phi$. i.e., $\frac{2(\alpha,\beta)}{(\beta,\beta)}\in\mathbb{Z}$ $\forall\alpha,\beta\in\Phi$.

③ If $\beta\in\Phi$ is a multiple of $\alpha\in\Phi$, then $\beta=\alpha$ or $\beta=-\alpha$.

Each $\alpha\in\Phi$ is called a root.

Ex. Φ is a root system $\Longrightarrow$ $\Phi^\vee=\{\alpha^\vee \mid \alpha\in\Phi\}$ is also a root system

Pf: ③ √ easy. ② $(\alpha^\vee,(\beta^\vee)^\vee)=(\alpha^\vee,\beta)\in\Phi$

① $$r_{\alpha^\vee}(\beta^\vee) = r_\alpha(\beta^\vee) = r_\alpha\left(\frac{2\beta}{(\beta,\beta)}\right) = \frac{2r_\alpha(\beta)}{(\beta,\beta)}$$

$$\Phi^\vee \ni \left(r_\alpha(\beta)\right)^\vee = \frac{2r_\alpha(\beta)}{\left(r_\alpha(\beta), r_\alpha(\beta)\right)} = \frac{2r_\alpha(\beta)}{(\beta,\beta)} \quad //$$

More defn.

- Φ is <u>reducible</u> if it's the union of two proper, orthogonal subsets which are themselves root systems. Otherwise Φ is <u>irreducible</u> or <u>simple</u>.

- Φ is <u>simply-laced</u> if all roots have the same length.

  How does this tie into our familiar notion of simply-laced for Coxeter graphs?

Consequences of the root system axiom 1.

1. Let $\alpha, \beta \in \Phi$, $\beta \neq -\alpha$. Then $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}$.

Angles

($\langle \, , \rangle$ is not symm in the arguments)

Pf: $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = \frac{2 (\alpha, \beta)}{\langle \beta, \beta \rangle} \frac{2 \langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 4 \frac{\|\alpha\|^2 \|\beta\|^2 \cos^2\theta}{\|\alpha\|^2 \|\beta\|^2}$

$= 4 \cos^2\theta \in \mathbb{Z} \cap \{0, 1, 2, 3, 4\}$, where θ is the angle between α and β.

$4 \cos^2\theta = 4 \Rightarrow \cos\theta = \pm 1 \Rightarrow \alpha = \pm\beta$, $\rightarrow\leftarrow$, so $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}$.

In fact, if we assume $\|\beta\| \geq \|\alpha\|$, there are only finitely many possibilities. $\left(|\langle \beta, \alpha^\vee \rangle| = \frac{|2 (\beta, \alpha)|}{\|\alpha\|^2} \geq \langle \alpha, \beta^\vee \rangle = \frac{|2(\alpha, \beta)|}{\|\beta\|^2} \right)$

$\langle\alpha,\beta^\vee\rangle\langle\beta,\alpha^\vee\rangle$	$\langle\alpha,\beta^\vee\rangle$	$\langle\beta,\alpha^\vee\rangle$	θ	$\frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle}$	$r_\beta(\alpha)=\alpha-\langle\alpha,\beta^\vee\rangle\beta.$	
0	0	0	$\frac{\pi}{2}$	undetermined.	α	
1	1	1	$\frac{\pi}{3}$	1	$\alpha-\beta$	
1	-1	-1	$\frac{2\pi}{3}$	1	$\alpha+\beta$	$\alpha\overset{(3)}{-}\beta$
2	1	2	$\frac{\pi}{4}$	2	$\alpha-\beta$	
2	-1	-2	$\frac{3\pi}{4}$	2	$\alpha+\beta$	$\beta\overset{4}{\Rightarrow}\alpha$
3	1	3	$\frac{\pi}{6}$	3	$\alpha-\beta$	
3	-1	-3	$\frac{5\pi}{6}$	3	$\alpha+\beta$	$\beta\overset{6}{\Rrightarrow}\alpha$

Corollary: Let $\alpha,\beta\in\Phi$ and let θ be the angle between them. Assume $\|\beta\|\geq\|\alpha\|$

① If $\theta>\frac{\pi}{2}$ then $\alpha+\beta\in\Phi$ ② If $\theta<\frac{\pi}{2}$ then $\alpha-\beta\in\Phi$.

2. **Thm.** (a) Every root system has a base, that is, a set $\Sigma = \{\alpha_i \mid i \in I\}$

Base.

s.t. ① Σ is linearly ind. and span Φ.

② every $\beta \in \Phi$ can be written as $\beta = \sum_{i \in I} c_i \alpha_i$

s.t. either $\underbrace{c_i \geq 0 \ \forall i \in I}_{\text{positive roots. } \Phi^+}$ or $\underbrace{c_i \leq 0 \ \forall i \in I}_{\text{negative roots } \Phi^-}$.

Elements of Σ are called the simple roots.

Note: if $\alpha, \beta \in \Sigma$, then $\langle \alpha, \beta \rangle < 0$. i.e. $\theta_{\alpha,\beta} > \frac{\pi}{2}$. (see earlier corollary)

(b) One way to get a base: Choose $z \in V$ not perpendicular to any roots, let $\Phi_z^+ = \{\alpha \in \Phi : (\alpha, z) > 0\}$ and let $\Sigma = \{\alpha \in \Phi_z^+ : \alpha$ is not a sum of two elts in $R^+\}$. Then Σ is a base of Φ with $\Phi^+ = \Phi_z^+$. In particular, a base of Φ always exists but is not unique.

(c) Pick a base Σ of Φ and a simple root α_i. Write $s_i = r_{\alpha_i}$.

Then ① On Φ^+ s_i sends α_i to $-\alpha_i$

and permutes $\Phi^+ \setminus \{\alpha_i\}$.

② The Weyl gp of Φ, the gp W gen. by $\{r_\alpha : \alpha \in \Phi\}$, is generated by $\{s_i : i \in I\}$. In fact,

there are relations among them automatically.

simple reflections

long → short

W is a finite Coxeter gp with $\{s_i : i \in I\}$ as its Coxeter generators.

$s_i^2 = 1$. ✓

reflection.

$(s_i s_j)^{m_{ij}} = 1$.

ref ref

order $= \frac{2\pi}{2\gamma} = \begin{cases} 3 \\ 4 \\ 6 \end{cases}$

Dynkin (see table)

simple edge

double bond

triple bond

rotation by 2γ, γ : acute angle between $\mathbb{C}\alpha_i$, $\mathbb{C}\alpha_j$

Note: $W(\Phi) = W(\Phi^\vee)$.

③ From any choice of base Σ for Φ, we may define an $I \times I$ matrix C called the Cartan matrix of Φ, with $C_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$.

(So $C_{ii} = 2 \ \forall i \in I$ and $\{C_{ij}, C_{ji}\} \in \{\{-1,-1\}, \{-1,-2\}, \{-1,-3\}, \{0\}\}$)

Up to reordering of rows and columns, C is independent of the choice of the base, so it's well-defined.

④ It turns out that using the fact that a root system is finite by definition, we can classify all root systems by their Dynkin diagram or Cartan types. The types are: A_n, $\underbrace{B_n, C_n}_{\text{dual}}$, D_n, E_6, E_7, E_8, F_4, G_2.

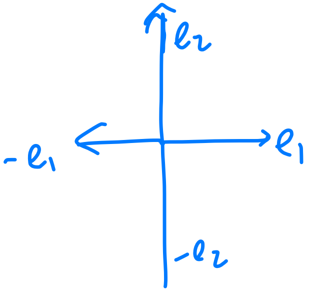
We can work out some possible root systems in $\mathbb{R}^2$ but not $\mathbb{R}$:

- $|\Sigma| \le \dim \mathbb{R}^2 = 2$. So Φ is 2-dim $\Longrightarrow |\Sigma| = 2$. say $\Sigma = \{\alpha, \beta\}$

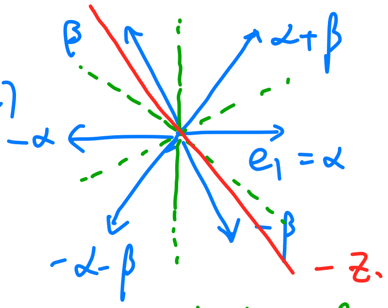
- Recall the possibilities assuming $\|\beta\| \ge \|\alpha\|$. May assume $\alpha = e_1$:

	$\langle\alpha,\beta\rangle\langle\beta,\alpha^\vee\rangle$	$\langle\alpha,\beta^\vee\rangle$	$\langle\beta,\alpha^\vee\rangle$	θ	$\frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle}$	$r_\beta(\alpha)=\alpha-\langle\alpha,\beta^\vee\rangle\beta$.	
(1)	0	0	0	$\frac{\pi}{2}$	undetermined.	α	
(2)	1	-1	-1	$\frac{2\pi}{3}$	1	$\alpha+\beta$	$\alpha \overset{(3)}{-} \beta$
(3)	2	-1	-2	$\frac{3\pi}{4}$	2	$\alpha+\beta$	$\beta \overset{<}{=}$
(4)	3	-1	-3	$\frac{5\pi}{6}$	3	$\alpha+\beta$	$\beta \overset{6}{\Rrightarrow} \alpha$

(1):



(2)

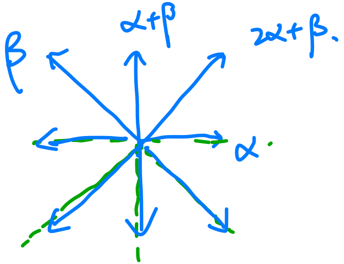


$\longrightarrow A_2$

$\rightarrow \Sigma_z = \{\alpha, \beta\}.$

green: the hyperplanes

(3)



$\longrightarrow B_2$

when $r = 2$, Φ lies in the hyperplane orthogonal to $(1,1,1)$ in $\mathbb{R}^3$. 2D!

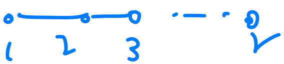
$$\cos\theta(\alpha_1,\alpha_2) = \frac{(\alpha_1,\alpha_2)}{\|\alpha_1\|\,\|\alpha_2\|} = \frac{-1}{2} \Rightarrow \theta = \frac{2}{3}\pi$$

similarly we can compute $\theta(\alpha,\beta)$ $\forall \alpha,\beta$.

↑ Then we can show $\Phi \cong Ex(2)$.

(4) G_2. Exercise.

(5) Exercise: Check that $\Phi = \{\pm(e_i - e_j) : 1 \le i < j \le r+1\}$ forms a root system in $\mathbb{R}^{r+1}$, and $\Sigma := \{ \underbrace{e_i - e_{i+1}}_{\alpha_i} \} \subset \Phi$ is a base of Φ. The Dynkin diagram of Φ is



, $\circlearrowleft \mathbb{R}^{r+1}$

and $S_i = r(\alpha_i)$ 'permutes e_i and e_{i+1}' so $W(\Phi) \cong S_{r+1}$.

$$s_i(e_j) = e_j - \langle e_j, e_i - e_{i+1} \rangle (e_i - e_{i+1}) = \begin{cases} e_j & \text{if } j \notin \{i, i+1\} \\ e_{i+1} & \text{if } j = i \\ e_i & \text{if } j = i+1. \end{cases}$$

Part II. Weights

Definition 1.

- Given a root system Φ in an Euclidean space V, a weight lattice is a lattice Λ spanning V s.t. $\Phi \subset \Lambda$ and $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ $\forall \lambda \in \Lambda, \alpha \in \Phi$.

  Elements of Λ are called weight.

- A wt. lattice $\Lambda \subset V$ is called semisimple if Φ spans V.

(the condition is not about Λ per se! Root systems coming from Lie algebras are often/can often be assumed to be semisimple.)

"This is equivalent to assuming that the root lattice Λ_{root} spanned by Φ has finite codimension in Λ." — there will be examples.

— There's a partial order on Λ: for $\lambda, \mu \in \Lambda$. write

$$\lambda \geq \mu \quad \text{if} \quad \lambda - \mu = \sum_{i \in I} c_i \alpha_i \quad \text{where } c_i \geq 0 \ \forall i \in I.$$

— (Dominant wts) Let $\Lambda_+ = \{ \lambda \in \Lambda : \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ \forall i \in I \}$

Elts of Λ_+ are called dominant wts. $\lambda \in \Lambda_+$ is strictly dominant if $\langle \lambda, \alpha_i^\vee \rangle > 0 \ \forall i \in I$.

— (Fundamental wts) $\forall i \in I$, $\exists\, \varpi_i \in V$ s.t.

fundamental wts

$$(*) \quad \langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}.$$

If Φ is semisimple (span V), $(*)$ determines ϖ_i ; otherwise there are choices for ϖ_i.

(bilinear form argument)

Note: fundament wts may not be in the wt lattice !

— Assuming $\Phi \subset \Lambda \subset V$ is semisimple. Then the fundamental weights generate a lattice Λ_{sc} that contains Λ_{root} as a sublattice of finite index. We have

$$\Lambda_{sc} \supseteq \Lambda \supseteq \Lambda_{root}.$$

$$\searrow$$

$$\lambda \in \Lambda,\ \langle \lambda, \alpha_i \rangle = c_i \geq 0 \quad \forall i \in I \implies \lambda = \sum c_i \bar{w}_i \in \Lambda_{sc}.$$

If $\Lambda = \Lambda_{root}$, we say Λ is of adjoint type.

(comes from adjoint rep)

If $\Lambda = \Lambda_{sc}$, i.e., if all fundamental wts are in the wt lattice. we say Λ is of simply connected type. (← can often enlarge Λ to be simply-connected)

(motivation?)

Examples.

(1) Type A_r. (BS Ex. 2.5. 'GL(r+1) repn')

$V = \mathbb{R}^{r+1}$. $\Phi = \{e_i - e_j \mid 1 \le i, j \le r+1\}$. $\Sigma = \{e_i - e_{i+1} \mid 1 \le i \le r\}$

not semisimple. $\Phi \perp e_1 + \cdots + e_{r+1}$.

can also pick $\bar{\omega}_i' = \bar{\omega}_i + (e_1 + \cdots + e_{r+1})$.

$\Phi^+ = \{e_i - e_j \mid 1 \le i < j \le r+1\}$.

Take $\Lambda = \mathbb{Z}^{r+1} = \{\underset{\substack{\shortparallel \\ \lambda}}{(\lambda_1, \dots, \lambda_{r+1})} : \lambda_i \in \mathbb{Z}\ \forall i\}$

Note: $\langle \lambda, (e_i - e_j)^\vee \rangle = \langle \lambda, e_i - e_j \rangle = \lambda_i - \lambda_j. \in \mathbb{Z}\ \forall \lambda \in \Lambda.$

and of course Λ span V.

$\lambda \in \Lambda^+ \iff (\lambda, e_i - e_{i+1}) \ge 0\ \forall i \iff \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{r+1}$

fundamental wts: $\bar{\omega}_i = e_1 + \cdots + e_i$ satisfies $\langle \bar{\omega}_i, \alpha_j^\vee \rangle = \delta_{ij}$. so

we can pick $\bar{\omega}_i$ to be the fundamental wt.

(2) (Semisimplification of (1). Type A_r, $SL(r+1)$ version)

$V = \mathbb{R}^{r+1} / \mathbb{C}\langle(1, \dots, 1)\rangle$. Φ = Image of Φ from (1) in the quotient V.

$\Lambda :=$ image of Λ from (1) in the quotient.

(3). $(B_r, \ Spin(2r+1))$ $V = \mathbb{R}^r$, $\Phi = \{\underbrace{\pm e_i \pm e_j \mid}_{} i<j\} \cup \{\underbrace{\pm e_i}_{}\}$

not simply-laced

$\frac{\|\text{long}\|^2}{\|\text{short}\|^2} = 2$

$\Phi^+ = \{e_i \pm e_j \mid i<j\} \cup \{e_i\}$.

$\Sigma = \{e_1 - e_2, \dots, e_{r-1} - e_r, e_r\}$.

$\Lambda = \{\underset{!!}{(\lambda_1, \dots, \lambda_r)} \in \mathbb{Q}^r \mid 2\lambda_i \in \mathbb{Z} \ \forall i, \ 2\lambda_i$ are either all odd or all even$\}$

all in Λ, so simply-connected!

Note: $\langle \lambda, e_i^\vee \rangle = 2\lambda_i$, $\langle \lambda, e_i \pm e_j \rangle = \lambda_i \pm \lambda_j$. so $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \lambda \in \Lambda, \ \alpha \in \Phi$.

dominant: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$.

fund. wts: $\varpi_1 = e_1$, $\varpi_2 = e_1 + e_2$, $\dots$, $\varpi_{r-1} = e_1 + \dots + e_{r-1}$ $\varpi_r = \frac{1}{2}(e_1 + \dots + e_r)$.

Exercises. (a) Check Ex 2.8.

Φ as in $(B_r, \mathrm{Spin}(2r+1))$

$\Lambda = \Lambda_{\text{root}} = \mathbb{Z}^r$, not simply-connected.

(b) Check Ex 2.9. C_r. everything is dual to $(B_r, \mathrm{Spin}(2n+1))$.

(c) Compute the type B_r and C_r Weyl gps

(They should be the same and iso to the Coxeter gp of type B.)

Part II. Lie algebras origins of root systems and wts

— A Lie algebra over a field k is a k-vector space with a bilinear binary operation $[\,,\,]: L \times L \longrightarrow L$, $(x, \ ) \mapsto [x.y]$. s.t.

① $[x,x] = 0 \ \forall x \in X$ and

② $[x[yz]] + [y[zx]] + [z[xy]] = 0 \ \forall x.y.z \in L.$ (Jacobi Identity)

Prototypical examples. ⓐ $gl(n.k)$ or $gl(V)$ for a v.s. V.

inv. matrizes inv endomorphisms on V

$[\,,\,]$ given by commutation: $[x,y] = xy - yx$.

ⓑ $sl(n.k)$. or $sl(V)$. elts of ⓐ w/ trace 0. same $[\,.\,]$.

Note: $[x,y]$ has trace 0 if $x.y$ have trace 0.

③ Other subsets of $\mathfrak{gl}(n,k)$ or $\mathfrak{gl}(n(V)$, giving rise to Lie algebras of type B, C, D. in a 'natural way'. $\mathfrak{sl}(n,k)$ is considered 'type A'.

④ In fact every Lie algebra is isomorphic to a 'linear Lie algebra': i.e., a sub-Lie alg of $\mathfrak{gl}(V)$ for some V.

⑤ Most important example to keep in mind: (we'll assume $k = \mathbb{C}$)

$$\mathfrak{sl}_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+d=0 \right\} = \operatorname{Span}_{\mathbb{C}} \left\{ \underset{e}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{f}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{h}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \right\}$$

Note: $\{e,f,h\}$ is a basis of $\mathfrak{sl}_2$. ✓

$$[h,e]=2e, \quad [h,f]=-2f \quad [e,f]=h. \quad (e_{ij}e_{kl}=\delta_{jk}e_{il})$$

— A representation of L is the data of a v.s V with a map

$$\rho : L \longrightarrow \mathfrak{gl}(V) \quad \text{s.t.} \quad \rho([x,y]) = [\rho(x), \rho(y)] \text{, ie,}$$

$$\text{s.t} \quad \rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x) \quad \forall x, y \in L.$$

↘ 'morphism of Lie algebras'.

Eg: $\mathrm{ad} : L \longrightarrow \mathfrak{gl}(L) \quad \rho(x) = [x, -]$. → adjoint rep.

Exercise: Check that the Jacobi id. ensures ad is a rep.

Facts:

① Every 'simple' Lie algebra is made up of / gen by $\mathfrak{sl}_2$-triples $(e_\alpha, f_\alpha, h_\alpha)$ indexed by elts α of a set Φ which turn out to be a root system.

E.g. $\mathfrak{sl}_3(\mathbb{C}) = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} : a+e+i=0 \right\}$. $\longleftrightarrow$ A_2

$$h_\alpha = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}, \quad e_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad f_\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \alpha \leftrightarrow e_1 - e_2$$

$$h_\beta = \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \quad e_\beta = \begin{bmatrix} 0 & & \\ & 0 & 1 \\ & 0 & 0 \end{bmatrix}, \quad f_\beta = \begin{bmatrix} 0 & & \\ & 0 & 0 \\ & 1 & 0 \end{bmatrix} \qquad \beta \leftrightarrow e_2 - e_3$$

$$h_{\alpha+\beta} = \begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix}, \quad e_{\alpha+\beta} = \begin{bmatrix} 0 & & 1 \\ & & \\ 0 & & 0 \end{bmatrix}, \quad f_{\alpha+\beta} = \begin{bmatrix} 0 & & 0 \\ & & \\ 1 & & 0 \end{bmatrix} \qquad \alpha+\beta \leftrightarrow e_1 - e_3$$

② The collection $H = \{ h_\alpha : \alpha \in \Phi \}$ will form a maximal abelian subalgebra.

$\downarrow$

$[h_1, h_2] = 0 \quad \forall h_1, h_2 \in H.$

Moreover, it's known that given any repn $L \to \mathfrak{gl}(V)$ of L, elts of H must act diagonalizably and hence (by linear algebra) simultaneously diagonalizably.

This gives $V = \bigoplus_{\lambda \in H^*} V_\lambda$ where

$$V_\lambda = \{ v \in V : h \cdot v = \lambda(h) v \} \neq \phi.$$

We define each $\lambda \in H^*$ for which $V_\lambda \neq 0$ to be a wt and let Λ be the set of wts.

For the adjoint rep $ad : L \to gl(L)$, the weights are also called roots, and the set of roots will form a root system in our sense

③ So roots of a $\overset{(\text{simple})}{\text{Lie}}$ alg / wts of a Lie alg rep are elts of H^*.
H is finite dimensional and will carry a nondegenerate bilinear form, $(,)$

So we'll be able to identify H with H^*.

killing form

H^* will correspond to our ambient space V for root systems and the form will give rise to our inner product.

④ It turns out that any (f.d.) rep of L is determined by the action of the triple $\{e_\alpha, f_\alpha, h_\alpha\} =: sl_2(\alpha) \cong sl_2$

We have $sl_2(\alpha) \hookrightarrow L \circlearrowright V$, and the rep theory of sl_2 is known, and by this knowledge we know that if

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda \quad \text{where} \quad V_\lambda = \{v \in V,\ h.v = \lambda(h)v\}$$

for $\lambda \in H^*$, then $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$ $\forall \alpha \in \Phi$.

⑤ With the identification of H and H^*, it will turn out that

$\lambda(h_\alpha) = (\lambda, \alpha^\vee)$. so V fd $\Rightarrow$ $(\lambda, \alpha^\vee) \in \mathbb{Z}$ $\forall \alpha \in \Phi$ $\Rightarrow$ $\lambda \in \Lambda^+$.