MATH 4140. HOMEWORK 8
due Wednesday, March 31

Note: All numbered sections, exercises, theorems and definitions are from Erdmann–Holm.

(1) Read Chapter 4

(2) Exercise 4.1.

(3) Exercise 4.2.

(4) Exercise 4.6. (*Hint*: see the lecture notes of March 26.)

(5) Prove that if A is a commutative k-algebra, then the Jacobson radical $J(A)$ contains all nilpotent elements of A.

(6) Prove that for any k-algebra A, we have $J(A/J(A)) = 0$; you can invoke Theorem 4.23 if you want to.

(7) Let k be an infinite field. Let f be a polynomial in $k[x]$ and let $f = f_1^{a_1} \dots f_r^{a_r}$ be the unique decomposition of f into pairwise coprime irreducible factors $f_1, \dots, f_r$. Let $A = k[x]/\langle f \rangle$.
   (a) Find the maximal ideals of A.
   (b) Find the Jacobson radical of A.
   (c) Describe when A is semisimple ("A is semisimple if and only if ..."), and justify your description.
   (d) Prove that $k[x]$ is not semisimple.
   (e) Prove that the Jacobson radical $J(k[x])$ of $k[x]$ is zero.
   (f) By the previous two parts, the algebra $k[x]$ has trivial Jacobson radical but is not semisimple. Why does this not contradict Part (g) of Theorem 4.23?